Introduction: Many problems in various branches of mathematics, science and engineering often require cumbersome analytical computation of solving, plotting animating the graph of one dimensional wave equation. These equations are so difficult in many cases, even impossible to perform by hand and the need to solve these problems with fast and easy analytic computation, which has lead to the idea of using maple to extensively perform an analytical, graphical and animation of one-dimensional wave equations.

Historically, the problem of wave equations was studied by Jean le Round d’ Alembert using a vibrating string such as that of a musical instrument. In 1746, d’Alembert discovered the one dimensional wave equation which is hyperbolic partial differential equation. It typically concerns a time variable $t$ one or more partial variables $x_1,x_2,.......x_n$ and a scalar function $U(x_1,x_2,...,x_n)$ wave defined as below.

$$\nabla^2 u(x, t) = C^2 \left( \frac{\partial^2}{\partial x^2} U(x, t) \right)$$

Or in separation of variables form which is computed as

$> U[tt] = C^2 U[xx];$

$$U_{tt} = C^2 U_{xx}$$
This was derived in class for small amplitude vibrations of a uniform string under a constant tension. The equation alone does not specify a solution but a unique solution is usually obtained by setting a problem with further conditions, such as initial conditions and another important class of problems called boundary conditions.

Since the wave equation is a linear second order PDE, given ant twice-differentiable functions of a single variable called $f_1$ and $f_2$

> \[ f_1(x + ct) \]

And

> \[ f_2(-ct + x) \]

In 18th century d’Alembert noted that the plus and minus signs in $x + ct$ and $x - ct$ indicate the direction of waves as $f_2(x - ct)$ travels to the right then $f_1(x + ct)$ travels to the left.

> \[ U(x, t) = f_1(x + ct) + f_2(x - ct) \]

The wave function can be computed as below.

\[
U(x, t) = \frac{1}{2} f(x + ct) + f(x - ct) + \frac{1}{2c} \left( \int_{x-ct}^{x+ct} g(s) \, ds \right)
\]

Basic Features of Maple 18 Software

- It is fast in symbolic, numerical and interactive computation.
- It is accessible to large number of students and researchers.
- It is available for almost all operating systems.
- It is powerful programming language, intuitive syntax and easy to debug.
- It consists an extensive library of mathematical functions and specialized packages like `with(plots), with(PDEtools)` e.t.c.

Methodology

Suppose the ICs are $u(x, 0) = \frac{1}{2} e^{-x^2}$ and $u_t(x, 0) = e^{-x^2}$ for $c = 4$ use d'Alembert computation to find the wave function $u(x, t)$.

Solution

Start with the command `With(PDEtools)` on the worksheet and define the wave equation formula with any choice of abbreviation e.g WaveEq or Wav.
The initial displacement and initial velocity are 

\[ u(x, 0) = \frac{1}{2} e^{-x^2} \] and 

\[ u_t(x, 0) = e^{-x^2} \]

with the speed \( c = 4 \).

Compute the command as `pdsolve` which means to find the partial differential equation with d’Alembert formula.

\[ \text{pdsolve}\left( \{ \text{WaveEq}, u(x, 0) = f(x), D[2](u)(x, 0) = g(x) \} \right); \]

\[ u(x, t) = \frac{1}{2} \cdot \frac{1}{c} \left( \int_{0}^{\frac{-ct+x}{c}} g(xl) \, dxl + \int_{0}^{\frac{ct+x}{c}} g(xl) \, dxl \right) \]

Use dA (d’Alembert) to combine into single term

\[ dA := \text{IntegrationTools:-Combine}(\%); \]

\[ dA := u(x, t) \]

\[ = \frac{1}{2} \cdot \frac{1}{c} \left( \int_{0}^{\frac{-ct+x}{c}} g(xl) \, dxl + f(-ct+x) \cdot c \right) \]

\[ + f(ct+x) \cdot c \]

Use `eval` to evaluate the expression in dA

\[ \text{eval}\left( dA, \{ c = 4, f = \left( x \rightarrow \frac{1}{2} \cdot \exp(-x^2) \right), g = \left( x \rightarrow \exp(-x^2) \right) \} \right) \]

\[ u(x, t) = \frac{1}{8} \int_{4t+x}^{4t+x} e^{-xl^2} \, dxl + \frac{1}{4} \cdot e^{-(-4t+x)^2} \]

\[ + \frac{1}{4} \cdot e^{-(4t+x)^2} \]

Use the command `value(%)` to evaluate the inert function of dA by defining it as `dA1` or with any other abbreviation.

\[ dA1 := \text{value}(\%); \]

\[ dA1 := u(x, t) = \frac{1}{16} \sqrt{\pi} \cdot \text{erf}(4t + x) + \frac{1}{16} \sqrt{\pi} \cdot \text{erf}(4t - x) + \frac{1}{4} \cdot e^{-(-4t+x)^2} + \frac{1}{4} \cdot e^{-(4t+x)^2} \]
Method of Plotting Graph f(x)

When \( f := x \rightarrow \frac{1}{2} \cdot \exp\left(-x^2\right); \)

\( f := x \rightarrow \frac{1}{2} \, e^{-x^2} \)

\( \text{plot}(f(x), x = -20 .. 20); \)

\[ \textbf{g} \ := \ x \rightarrow e^{-x^2} \]

\( \text{plot}(\text{g}(x), x = -20 .. 20); \)

\[ \textbf{f} \ := \ x \rightarrow \frac{1}{2} \, e^{-x^2} \]

\[ \textbf{g} \ := \ x \rightarrow e^{-x^2} \]

METHOD OF PLOTTING GRAPH g(x)

When \( g := x \rightarrow \exp\left(-x^2\right); \)

(a) Then enter \( \text{plot}(g(x), x = -20 .. 20); \) on the work sheet.

(b) Immediately the graph appears, click on the graph.

(c) Move the mouse to \textit{change gridlines properties} on the standard toolbar and click.

(d) On \textit{Axis Gridlines Properties} with the \textit{Vertical} Option select \textit{Show Gridlines}, your choice of color and click \textit{apply}.

(e) Select the \textit{Horizontal} option and follow the same procedure

(f) Finally click the \textit{OK}.

\( \text{plot3d} \left( \cos(u(x, t)), x = -40 \cdot \Pi .. 150, t = -\frac{76}{\Pi} \cdots \frac{20}{\Pi}, \text{axes} = \text{boxed} \right) \)
Method of Plotting 3-D Graph Of Solution \( \cos u(x,t) \).

Since

\[
dA := u(x,t) = \frac{1}{16} \sqrt{\pi} \text{erf}(4t + x) + \frac{1}{16} \sqrt{\pi} \text{erf}(4t - x) + \frac{1}{4} + \frac{1}{4} e^{-(4t + x)^2}
\]

(a) Then enter

\[
\text{plot3d}(\cos(u(x,t)), x = -40 \cdot \text{Pi}..150, t = -\frac{76}{\text{Pi}}..\frac{20}{\text{Pi}}, \text{axes = boxed}),
\]

on the work sheet.

(b) Immediately the graph appears, move the cursor and click on the graph.

(c) Move the mouse to **change axis properties** on the standard toolbar and left click.

(d) On **Axis Properties** with the X-axis Option select **Log mode**, your choice of **color** and click **apply**.

(e) Select the Y and Z-axis option and follow the same procedure

(f) Finally click the **OK** as the graph changes to the object below

Animation of 3-D Graph

Apply d’Alembert formula to form the animation and 3-D graph of a wave function given by the initial conditions \( u(x, 0) = \frac{1}{2} e^{-x^2}, ut(x,0) = 0 \) with \( c = 4 \).

> with(PDEtools):

> \( > \text{WaveEq} := \text{diff}(u(x,t), t, t) = c^2 \cdot \text{diff}(u(x,t), x, x); \)

> \( > \text{WaveEq} := \frac{\partial^2}{\partial t^2} u(x, t) = c^2 \left( \frac{\partial^2}{\partial x^2} u(x, t) \right) \)

> \( > \text{pdsolve}( \{ \text{WaveEq}, u(x, 0) = f(x), D[2](u)(x, 0) = g(x) \} ); \)

\[
\begin{align*}
\text{u}(x, t) &= \frac{1}{2} \frac{1}{c} \left( f(-ct + x) c + f(ct + x) c - \left( \int_{0}^{\frac{ct+x}{c}} g(x) \, dx \right) \right) \\
&+ \left( \frac{ct+x}{c} \right) g(x) \, dx
\end{align*}
\]

> \( dA := \text{IntegrationTools:-Combine}(); \)
\[ dA := u(x, t) \]

\[
= \frac{1}{2} \left( -\int_{c t+x}^{c t+x} g(xl) \, dxl \right) + f(-c t+x) \, c + f(c t+x) \bigg/ c
\]

> eval \( (dA, \{ c = 4, f = (x \rightarrow \frac{1}{2}, \exp(-x^2)), g = 0 \}) \);

\[ u(x, t) = \frac{1}{8} \int_{4 t+x}^{-4 t+x} 0 \, dxl + \frac{1}{4} \, e^{-(-4 t+x)^2} + \frac{1}{4} \, e^{-(4 t+x)^2} \]

> \( dA := \text{value}() \);

\[ dA := u(x, t) = \frac{1}{4} \, e^{-(4 t+x)^2} + \frac{1}{4} \, e^{-(4 t+x)^2} \]

with(plots) :

\[
\text{plot3d} \left( \frac{1}{4} \, e^{-(4 t+x)^2} + \frac{1}{4} \, e^{-(4 t+x)^2}, x = -5..5, t = -5..5 \right);
\]

\[ \text{Fig. 5} \]

**METHOD OF PLOTTING 3-D GRAPH OF SOLUTION \( u(x, t) \).**

When

\[ dA := u(x, t) = \frac{1}{4} \, e^{-(4 t+x)^2} + \frac{1}{4} \, e^{-(4 t+x)^2} \]

then

(a) Enter \( \text{plot3d} \left( \frac{1}{4} \, e^{-(4 t+x)^2} + \frac{1}{4} \, e^{-(4 t+x)^2}, x = -5..5, t = -5..5 \right) \); on the work sheet.

(b) Immediately the graph appears, click on the graph.

(c) Move the mouse to **change axis properties** on the standard toolbar and left click.
(d) On **Axis Properties** with the **X-axis** Option select **Log mode**, your choice of **color** and click **apply**.

(e) Select the **Y** and **Z-axis** option and follow the same procedure

(f) Finally click the **OK** as the graph changes to the object below.

\[
\text{plots}_{\text{animate}}\left(\text{plots}_{\text{complexplot3d}}\left(\frac{1}{4} e^{-(-4 t+x)^2} + \frac{1}{4} e^{(-4 t+x)^2}, x = -15 .. -15 + 1, \text{labels} \right), t = 0 .. 10\right);
\]

![Graph of equation](image)
Move the cursor to the graph, right click the mouse and select animation and play on the menu drop down list.
Plot the 3-D graph of the pulse begin at \( t = 0..10 \) when \( u = (x, 0) = e^{-\left(x-3\right)^2} \) and \( u_t = (x, 0) = 0 \) at \( c=4 \).

> restart;
> with(PDEtools):
> WaveEq := diff(u(x, t), t, t) = c^2 \cdot \text{diff}(u(x, t), x, x);
> WaveEq := \frac{\partial^2}{\partial t^2} u(x, t) = c^2 \left( \frac{\partial^2}{\partial x^2} u(x, t) \right)
> pdsolve( \{ WaveEq, u(x, 0) = f(x), D[2](u)(x, 0) = g(x) \} );
> u(x, t) = \frac{1}{2} \left[ f(-ct+x) c + f(ct+x) c - \int_{0}^{ct+x} g(xl) \, dxl \right] + \int_{0}^{ct+x} g(xl) \, dxl
> dA := IntegrationTools:-Combine(\%);
> dA := u(x, t)
= \frac{1}{2} \left[ f(-ct+x) c + f(ct+x) c \right] \int_{ct+x}^{-ct+x} g(xl) \, dxl + f(-ct+x) c + f(ct+x) c
> eval(\{ dA, \{ c=4, f = (x->\frac{1}{2} \cdot \exp - (x - 3)^2 ), g = 0 \} \});
> u(x, t) = -\frac{1}{8} \int_{4t+x}^{-4t+x} 0 \, dxl + \frac{1}{2} \exp - \frac{1}{2} \left( -4t + x - 3 \right)^2
- \frac{1}{2} \left( 4t + x - 3 \right)^2
> dAl := value(\%);
> dAl := u(x, t) = \frac{1}{2} \exp - \frac{1}{2} \left( -4t + x - 3 \right)^2 - \frac{1}{2} \left( 4t + x - 3 \right)^2

with(plots);
Using Maple 18 software in solving one-dimensional wave equation is an exposure and development to the computation of d’Alembert.

**SUMMARY**

This project work shows the procedure in solving one-dimensional wave equation with d’Alembert formula using Maple 18 software.

1. The word **Restart** is used to clear the internal memory of previous solving that may affect the new solving.
2. Wave formula is defined on the worksheet as

   \[ WaveEq := \frac{\partial^2}{\partial t^2} u(x, t) = c^2 \left( \frac{\partial^2}{\partial x^2} u(x, t) \right) \]
The calling sequence for computing d'Alembert formula is

\[ pdsolve( \{ \text{WaveEq, } u(x, 0) = f(x), D[2] (u)(x, 0) = g(x) \} ); \]

\[
u(x, t) = \frac{1}{2} \frac{1}{c} \left[ f(-ct + x) - f(ct + x) + \int_{0}^{ct + x} g(xl) \, dxl \right]
\]

(4) The calling sequence for solving with d'Alembert formula is

\[ eval( res, \{ c = 4, f = \left( x \rightarrow \frac{1}{2} \cdot \exp(-x^2) \right), g = \left( x \rightarrow \exp(-x^2) \right) \} ); \]

REFERENCE

- infoClearinghouse.com
- Mathematics and Physics Department University of Sonora.