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Original Research Article

FIXED POINT THEOREM AND ITS APPLICATION

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Abstract*:* In this paper, we introduce fixed point theorem in ordered Banach spaces closed convex normal cone K and using mixed increasing operator and also give a theorem inreal Banach space ordered by a closed convex normal cone Kand using mixed increasing operator with two linear continuous operators $L, S : X \to X$ with $L(K) \subset K$, $S(K) \subset K$ and$r(L + S) < \frac{1}{2}$, An example is given to illustrate the main result. Finally, we give applications of our results to solve a class of volterra type integral equation.

Keywords: - Operators, Banach Spaces, Mixed Increasing Operators, Fixed Points, Volterra type integral equations.

Introduction*:* Fixed point theorem for increasing operators in Banach spaces are extensively investigated and founded a range of application to differentialequation.

In H. Amann[1] gave a survey over some of the most important methods and results of nonlinear functional analysis in ordered Banach spaces. By means of iterative techniques and by using topological tools, fixed point theorems for completely continuous maps in ordered Banach spaces are deduced, and particular attention is paid to the derivation of multiplicity results. Moreover, solvability and bifurcation problems for fixed point equations depending nonlinearly on a real parameter are investigated.

Some existence theorems of the coupled fixed points for both continuous and discontinuous operators given by Dajun Guo [2] and then offer some applications to the initial value problems of ordinary differential equations with discontinuous right-hand sides.

Existence theorems of coupled fixed points for mixed monotone operators have been considered by several authors S.S. Chang [3],Yongzhuo[4], K.Deimling[5], D.J. Guo [6]. In S. S. Chang[7], study the existence problems of coupled fixed points for two more general classes of mixed monotone operators and apply main result to

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show the existence of coupled fixed points for a class of nonlinear integral equations.

Let X be a real Banach Space and K be a closed convex cone in X. First, let us recall that $K \subset X$ is called a closed convex cone if K is closed and the following conditions hold:

(i) $K + K \subset K$

(ii) $tK \subset K$ for all $t \geq 0$

(iii) $K \cap (-K) = \{0\}$

A partial order "≤" can be induced by K by $x \leq y$ if and only if $y - x \in K$

If $x \leq y$, we denote $[x,y] = \{z \in X : x \leq z \leq y\}$

The closed convex cone K is said to be normal if there exists a constant N> 0 such that $0 \leq x \leq y$ implies that $\|x\| \leq N\|y\|$.

In this paper,we prove some fixed point theorem for mixed increasing operator in ordered Banach spaces and also give an application to a class of volterra type integral equation.

Preliminaries

Definition 2.1:- An operator $A: M \subset X \to X$ is said to increasing if $x, y \in M, x \leq y$ implies that $Ax \leq Ay$

Definition 2.2: – An operator $F: M \times M \to X$ is said to be increasing if , for $x_1, x_2, y_1, y_2 \in M$, $x_1 \leq x_2$ and $y_2 \leq y_1$ imply that $F(x_1, y_1) \leq F(x_2, y_2)$.

Definition 2.3:–Let $F: M \times M \to X$ be an operator. We say that $(x^*, y^*) \in M \times M$ is a coupled fixed point of F if $F(x^*, y^*) = x^*$ and $F(y^*, x^*) = y^*$ and a point $x^* \in M$ is called a fixed point of F if $F(x^*, x^*) = x^*$.

Main Result

Theorem 3.1:- Let $(X, \leq)$ be a real Banach space ordered by a closed convex cone K. Let $x_0, y_0 \in X, x_0 \leq y_0$ and $M = [x_0, y_0]$. Suppose that $F: M \times M \to X$ is a mixed increasing operator satisfying the following conditions:

(i) For any $a, b, c, d \in M$, $a \leq b, d \leq c$ implies that

$$\|F(a,b) - F(c,d)\| \leq \frac{\alpha}{2}\{\|d - F(d,c)\| + \|b - F(d,c)\|\} + \frac{\beta}{2}\{\|F(a,b) - c\| + \|d - b\|\} \quad \text{....(3.1)}$$

Where $\alpha, \beta \geq 0$ such that $\alpha + \beta < \frac{1}{2}$

(ii) $x_0 \leq F(x_0, y_0)$ and $y_0 \leq F(y_0, x_0)$ ...(3.2)

Then F has a fixed a unique fixed point $x^* \in M$

Proof :- Define $\{x_n\}, \{y_n\}$ as follows :

$$x_n \leq F(x_{n-1}, y_{n-1}), \quad y_n \leq F(y_{n-1}, x_{n-1}) \quad \text{...(3.3)}$$

We claim that

$$x_0 \leq x_1 \leq \ldots \leq x_n \leq \ldots y_n \leq \ldots y_1 \leq y_0 \quad \text{...(3.4)}$$

In fact, for $n = 1$, since F is mixed increasing, it follows from (3.2) that

$x_0 \leq x_1 \leq y_1 \leq y_0$

Suppose that for $n = k (\geq 1)$, we have

$$x_0 \leq x_1 \leq \cdots \leq x_{k-1} \leq x_k \leq y_k \leq y_{k-1} \leq \cdots y_1 \leq y_0 \quad \text{...(3.5)}$$

Since F is mixed increasing,it follows from (3.5) that

$$x_k = F(x_{k-1}, y_{k-1}) \leq F(x_{k-1}, y_{k-1}) = x_{k+1}$$
$$y_{k+1} = F(y_{k-1}, x_{k-1}) \leq F(y_{k-1}, x_{k-1}) = y_{k+1}$$
$$x_{k+1} = F(x_k, y_k) \leq F(y_k, x_k) = y_{k+1} \quad \text{...(3.6)}$$

Combining (3.5) and (3.6), we get

$x_0 \leq x_1 \leq \cdots \leq x_{k+1} \leq y_{k+1} \leq \cdots y_1 \leq y_0$

By induction we conclude that (3.4) holds.

Now we show that for all $n \geq 1$

$$\|x_{n+1} - x_n\| \leq \left(\frac{2(\alpha+\beta)}{2-\beta}\right)^n \frac{1}{2}\{\|x_1 - x_0\| + \|y_0 - y_1\|\}$$

$$\|y_{n+1} - y_n\| \leq \left(\frac{2(\alpha+\beta)}{2-\beta}\right)^n \frac{1}{2}\{\|x_1 - x_0\| + \|y_0 - y_1\|\}$$

$$\|y_n - x_n\| \leq (\alpha+\beta)^n\{\|y_0 - x_0\|\} \quad \text{... (3.7)}$$

$$\|x_2 - x_1\| = \|F(x_1, y_1) - F(x_0, y_0)\|$$

$$\leq \frac{\alpha}{2}\{\|y_0 - F(y_0,x_0)\| + \|y_1 - F(y_0,x_0)\| + \|x_1 - x_0\|\}$$
$$+ \frac{\beta}{2}\{\|F(x_1,y_1) - x_0\| + \|y_0 - y_1\|\}$$
$$= \frac{\alpha}{2}\{\|y_0 - y_1\| + \|y_1 - y_1\| + \|x_1 - x_0\|\} + \frac{\beta}{2}\{\|x_2 - x_0\| + \|y_0 - y_1\|\}$$
$$= \frac{\alpha}{2}\{\|y_0 - y_1\| + \|x_1 - x_0\|\} + \frac{\beta}{2}\{\|x_2 - x_0\| + \|y_0 - y_1\|\}$$
$$\leq \frac{\alpha}{2}\{\|y_0 - y_1\| + \|x_1 - x_0\|\} + \frac{\beta}{2}\{\|x_2 - x_1\| + \|x_1 - x_0\| + \|y_1 - y_0\|\}$$
$$= \frac{\alpha}{2}\{\|y_0 - y_1\| + \|x_1 - x_0\|\} + \frac{\beta}{2}\{\|x_1 - x_0\| + \|y_0 - y_1\|\} + \frac{\beta}{2}\|x_2 - x_1\|$$
$$\|x_2 - x_1\| - \frac{\beta}{2}\|x_2 - x_1\| \leq \left(\frac{\alpha}{2} + \frac{\beta}{2}\right)\{\|y_0 - y_1\| + \|x_1 - x_0\|\}$$
$$\left(1 - \frac{\beta}{2}\right)\|x_2 - x_1\| \leq \left(\frac{\alpha}{2} + \frac{\beta}{2}\right)\{\|y_0 - y_1\| + \|x_1 - x_0\|\}$$
$$\|x_2 - x_1\| \leq \left(\frac{\alpha+\beta}{2-\beta}\right)\{\|x_1 - x_0\| + \|y_0 - y_1\|\} \qquad \dots(3.8)$$

Now,

$$\|y_2 - y_1\| = \|F(y_1,x_1) - F(y_0,x_0)\|$$
$$\leq \frac{\alpha}{2}\{\|x_0 - F(x_0,y_0)\| + \|x_1 - F(x_0,y_0)\| + \|y_1 - y_0\|\}$$
$$+ \frac{\beta}{2}\{\|F(y_1,x_1) - y_0\| + \|x_0 - x_1\|\}$$
$$= \frac{\alpha}{2}\{\|x_0 - x_1\| + \|x_1 - x_1\| + \|y_1 - y_0\|\} + \frac{\beta}{2}\{\|y_2 - y_0\| + \|x_0 - x_1\|\}$$
$$= \frac{\alpha}{2}\{\|x_0 - x_1\| + \|y_1 - y_0\|\} + \frac{\beta}{2}\{\|y_2 - y_0\| + \|x_0 - x_1\|\}$$
$$\leq \frac{\alpha}{2}\{\|x_0 - x_1\| + \|y_1 - y_0\|\} + \frac{\beta}{2}\{\|y_2 - y_1\| + \|y_1 - y_0\| + \|x_0 - x_1\|\}$$
$$= \frac{\alpha}{2}\{\|x_0 - x_1\| + \|y_1 - y_0\|\} + \frac{\beta}{2}\{\|y_1 - y_0\| + \|x_0 - x_1\|\} + \frac{\beta}{2}\|y_2 - y_1\|$$
$$\left(1 - \frac{\beta}{2}\right)\|y_2 - y_1\| \leq \left(\frac{\alpha}{2} + \frac{\beta}{2}\right)\{\|x_0 - x_1\| + \|y_1 - y_0\|\}$$
$$\|y_2 - y_1\| \leq \left(\frac{\alpha+\beta}{2-\beta}\right)\{\|x_1 - x_0\| + \|y_0 - y_1\|\} \qquad \dots(3.9)$$

Now,

$$\|y_1 - x_1\| = \|F(y_0,x_0) - F(x_0,y_0)\|$$
$$\leq \frac{\alpha}{2}\{\|y_0 - F(y_0,x_0)\| + \|x_0 - F(y_0,x_0)\| + \|y_0 - x_0\|\}$$
$$+ \frac{\beta}{2}\{\|F(y_0,x_0) - x_0\| + \|y_0 - x_0\|\}$$
$$= \frac{\alpha}{2}\{\|y_0 - y_1\| + \|x_0 - y_1\| + \|y_0 - x_0\|\} + \frac{\beta}{2}\{\|y_1 - x_0\| + \|y_0 - x_0\|\}$$
$$= \frac{\alpha}{2}\{\|y_0 - y_1\| + \|x_0 - y_1\| + \|y_0 - x_0\|\} + \frac{\beta}{2}\{\|y_1 - x_0\| + \|y_0 - x_0\|\}$$
$$\leq \frac{\alpha}{2}\{\|y_0 - x_0\| + \|y_0 - x_0\|\} + \frac{\beta}{2}\{\|y_1 - x_0\| + \|y_0 - x_0\|\}$$

Since $y_0 \geq y_1$

$$\leq \frac{\alpha}{2}\{\|y_0 - x_0\| + \|y_0 - x_0\|\} + \frac{\beta}{2}\{\|y_0 - x_0\| + \|$$
$$\|y_1 - x_1\| \leq (\alpha+\beta)\{\|y_0 - x_0\|\}$$

Again,

$$\|x_3 - x_2\| = \|F(x_2,y_2) - F(x_1,y_1)\|$$
$$\leq \frac{\alpha}{2}\{\|y_1 - F(y_1,x_1)\| + \|y_2 - F(y_1,x_1)\| + \|x_2 - x_1\|\}$$
$$+ \frac{\beta}{2}\{\|F(x_2,y_2) - x_1\| + \|y_1 - y_2\|\}$$
$$= \frac{\alpha}{2}\{\|y_1 - y_2\| + \|y_2 - y_2\| + \|x_2 - x_1\|\} + \frac{\beta}{2}\{\|x_3 - x_1\| + \|y_1 - y_2\|\}$$
$$= \frac{\alpha}{2}\{\|y_1 - y_2\| + \|x_2 - x_1\|\} + \frac{\beta}{2}\{\|x_3 - x_1\| + \|y_1 - y_2\|\}$$
$$\leq \frac{\alpha}{2}\{\|y_1 - y_2\| + \|x_2 - x_1\|\} + \frac{\beta}{2}\{\|x_3 - x_2\| + \|x_2 - x_1\| + \|y_1 - y_2\|\}$$
$$= \frac{\alpha}{2}\{\|y_1 - y_2\| + \|x_2 - x_1\|\} + \frac{\beta}{2}\{\|x_2 - x_1\| + \|y_1 - y_2\|\} + \frac{\beta}{2}\|x_3 - x_2\|$$
$$\|x_3 - x_2\| - \frac{\beta}{2}\|x_3 - x_2\| \leq \left(\frac{\alpha}{2} + \frac{\beta}{2}\right)\{\|y_1 - y_2\| + \|x_2 - x_1\|\}$$
$$\left(1 - \frac{\beta}{2}\right)\|x_3 - x_2\| \leq \left(\frac{\alpha}{2} + \frac{\beta}{2}\right)\{\|y_1 - y_2\| + \|x_2 - x_1\|\}$$
$$\|x_2 - x_1\| \leq \left(\frac{\alpha+\beta}{2-\beta}\right)\{\|y_1 - y_2\| + \|x_2 - x_1\|\}$$

From (3.8) and (3.9)

$$\|x_3 - x_2\| \leq \frac{(\alpha+\beta)}{(2-\beta)}\left\{\frac{(\alpha+\beta)}{(2-\beta)}\{\|x_1 - x_0\| + \|y_0 - y_1\|\} + \frac{(\alpha+\beta)}{(2-\beta)}\{\|x_1 - x_0\| + \|y_0 - y_1\|\}\right\}$$

$$\|x_3 - x_2\| \le \frac{(\alpha+\beta)}{(2-\beta)}\frac{(\alpha+\beta)}{(2-\beta)}\{\{\|x_1 - x_0\| + \|y_0 - y_1\|\} + \{\|x_1 - x_0\| + \|y_0 - y_1\|\}\}$$

$$\|x_3 - x_2\| \le \left(\frac{\alpha+\beta}{1-\beta}\right)^2 2\{\|x_1 - x_0\| + \|y_0 - y_1\|\} \quad \text{....(3.11)}$$

Now,

$$\|y_3 - y_2\| = \|F(y_2, x_2) - F(y_1, x_1)\|$$

$$\le \frac{\alpha}{2}\{\|x_1 - F(x_1, y_1)\| + \|x_2 - F(x_1, y_1)\| + \|y_2 - y_1\|\}$$
$$+\frac{\beta}{2}\{\|F(y_2, x_2) - y_1\| + \|x_1 - x_2\|\}$$
$$= \frac{\alpha}{2}\{\|x_1 - x_2\| + \|x_2 - x_2\| + \|y_2 - y_1\|\} + \frac{\beta}{2}\{\|y_3 - y_1\| + \|x_1 - x_2\|\}$$
$$= \frac{\alpha}{2}\{\|x_1 - x_2\| + \|y_2 - y_1\|\} + \frac{\beta}{2}\{\|y_3 - y_1\| + \|x_1 - x_2\|\}$$
$$\le \frac{\alpha}{2}\{\|x_1 - x_2\| + \|y_2 - y_1\|\} + \frac{\beta}{2}\{\|y_3 - y_2\| + \|y_2 - y_1\| + \|x_1 - x_2\|\}$$
$$= \frac{\alpha}{2}\{\|x_1 - x_2\| + \|y_2 - y_1\|\} + \frac{\beta}{2}\{\|y_2 - y_1\| + \|x_1 - x_2\|\} + \frac{\beta}{2}\|y_3 - y_2\|$$
$$\left(1 - \frac{\beta}{2}\right)\|y_3 - y_2\| \le \left(\frac{\alpha}{2} + \frac{\beta}{2}\right)\{\|x_1 - x_2\| + \|y_2 - y_1\|\}$$

$$\|y_2 - y_1\| \le \left(\frac{\alpha+\beta}{2-\beta}\right)\{\|x_1 - x_2\| + \|y_2 - y_1\|\}$$

From (3.8) and (3.9)

$$\|y_3 - y_2\| \le \frac{(\alpha+\beta)}{(2-\beta)}\left\{\frac{(\alpha+\beta)}{(2-\beta)}\{\|x_1 - x_0\| + \|y_0 - y_1\|\} + \frac{(\alpha+\beta)}{(2-\beta)}\{\|x_1 - x_0\| + \|y_0 - y_1\|\}\right\}$$

$$\|y_3 - y_2\| \le \left(\frac{\alpha+\beta}{2-\beta}\right)^2 2\{\|x_1 - x_0\| + \|y_0 - y_1\|\} \quad \text{...(3.12)}$$

Now,

$$\|y_2 - x_2\| = \|F(y_1, x_1) - F(x_1, y_1)\|$$

$$\le \frac{\alpha}{2}\{\|y_1 - F(y_1, x_1)\| + \|x_1 - F(y_1, x_1)\| + \|y_1 - x_1\|\}$$
$$+\frac{\beta}{2}\{\|F(y_1, x_1) - x_1\| + \|y_1 - x_1\|\}$$
$$= \frac{\alpha}{2}\{\|y_1 - y_2\| + \|x_1 - y_2\| + \|y_1 - x_1\|\} + \frac{\beta}{2}\{\|y_2 - x_1\| + \|y_1 - x_1\|\}$$
$$= \frac{\alpha}{2}\{\|y_1 - y_2\| + \|x_1 - y_2\| + \|y_1 - x_1\|\} + \frac{\beta}{2}\{\|y_2 - x_1\| + \|y_1 - x_1\|\}$$
$$\le \frac{\alpha}{2}\{\|y_1 - x_1\| + \|y_1 - x_1\|\} + \frac{\beta}{2}\{\|y_2 - x_1\| + \|y_1 - x_1\|\}$$

Since $y_1 \ge y_2$

$$\le \frac{\alpha}{2}\{\|y_1 - x_1\| + \|y_1 - x_1\|\} + \frac{\beta}{2}\{\|y_1 - x_1\| + \|y_1 - x_1\|\}$$

$$\|y_2 - x_2\| \le (\alpha+\beta)\|y_1 - x_1\|$$

$$\|y_2 - x_2\| \le (\alpha+\beta)^2\|y_0 - x_0\|$$

Continue in this manner, we get

$$\|x_{n+1} - x_n\| \le \left(\frac{2(\alpha+\beta)}{2-\beta}\right)^n \frac{1}{2}\{\|x_1 - x_0\| + \|y_0 - y_1\|\}$$

$$\|y_{n+1} - y_n\| \le \left(\frac{2(\alpha+\beta)}{2-\beta}\right)^n \frac{1}{2}\{\|x_1 - x_0\| + \|y_0 - y_1\|\}$$

$$\|y_n - x_n\| \le (\alpha+\beta)^n\{\{\|y_0 - x_0\|\}\}$$

Since $\alpha + \beta < \frac{1}{2}$ implies that $\{x_n\}$ and $\{y_n\}$ are two Cauchy sequences with same limit.

Let $\lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n = x^*$. Since K is closed, it is easy to know that

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$$x_n \le x^* \le y_n \quad \text{...(3.14)}$$

For all $n \ge 0$. Thus we have $x^* \in M$. it follows from 1 and 8 that

$\|x_{n+1} - F(x^*, x^*)\| = \|F(x_n, y_n) - F(x^*, x^*)\|$

Implies that

$$x^* = \lim_{n\to\infty} x_{n+1} = F(x^*, x^*)$$

This implies that $x^* \in M$ is a fixed point of F.

Now, we show that x^* is the unique fixed point of F.

Let $\bar{x} \in M$ be another fixed point of F. Since F is mixed increasing, we know that $x_n \le x^* \le y_n$ for all $n \ge 0$.

Since K is closed, it is easy to that $x^* \le \bar{x} \le x^*$. thus $x^* = \bar{x}$ this implies that $\bar{x}$ is the unique fixed point of F.

Theorem 3.2:- Let $(X, \leq)$ be a real Banach space ordered by a closed convex normal cone K. Let $x_0, y_0 \in X, x_0 \leq y_0$ and $M = [x_0, y_0]$. Suppose that $F: M \times M \to X$ is a mixed increasing operator satisfying the following conditions:

(i) There exists two linear continuous operators $L, S : X \to X$ with L(K)$\subset K$, S(K)$\subset K$ and $r(L+S) < \frac{1}{2}$ such that

$$F(a,b) - F(c,d) \leq L\{d - b + a - c\} + S\{F(a,b) - c + d - b\} \quad \text{....(3.15)}$$

For any $a, b, c, d \in M$, $a \leq c$, $d \leq b$, Where $L, S \geq 0$, $r(L+S)$ denotes the spectral radius of $L+S$

(ii) $x_0 \leq F(x_0, y_0)$ and $y_0 \leq F(y_0, x_0)$

Proof:- Define $\{x_n\}$, $\{y_n\}$ as follows :

$x_n \leq F(x_{n-1}, y_{n-1})$, $y_n \leq F(y_{n-1}, x_{n-1})$

As proved in theorem 3.1, we have

$$x_0 \leq x_1 \leq \ldots \leq x_n \leq \ldots y_n \leq \ldots y_1 \leq y_0 \quad \text{...(3.16)}$$

Since S, L are two linear continuous operators with $L(K) \subset K$ and $(K) \subset K$,

Let $x \leq y$ them $y - x \in M$. Since L(K)$\subset K$, we have $L(y-x) = Ly - Lx \in K$. This implies $L(x) \leq L(y)$ whenever $x \leq y$.

It follows that S , L are increasing.

Now,

$$\begin{aligned} x_2 - x_1 &= F(x_1, y_1) - F(x_0, y_0) \\ &\leq L\{y_0 - y_1 + x_1 - x_0\} + S\{F(x_1, y_1) - x_0 + y_0 - y_1\} \\ &= L\{y_0 - y_1 + x_1 - x_0\} + S\{x_2 - x_0 + y_0 - y_1\} \\ &= L\{y_0 - y_1 + x_1 - x_0\} + S\{x_2 - x_0 + y_0 - y_1\} \\ &= L\{y_0 - y_1 + x_1 - x_0\} + S\{x_2 - x_1 + x_1 - x_0 + y_0 - y_1\} \\ &= L\{y_0 - y_1 + x_1 - x_0\} + S\{x_1 - x_0 + y_0 - y_1\} + S(x_2 - x_1) \end{aligned}$$

$$(1-S)\,(x_2 - x_1) \leq (L+S)\,\{y_0 - y_1 + x_1 - x_0\}$$

$$(x_2 - x_1) \leq \left(\frac{L+S}{1-S}\right)\{y_0 - y_1 + x_1 - x_0\}$$

$$\begin{aligned} y_2 - y_1 &= F(y_1, x_1) - F(y_0, x_0) \\ &\leq L\{x_0 - x_1 + y_1 - y_0\} + S\{F(y_1, x_1) - y_0 + x_0 - x_1\} \\ &\leq L\{x_0 - x_1 + y_1 - y_0\} + S\{y_2 - y_0 + x_0 - x_1\} \\ &= L\{x_0 - x_1 + y_1 - y_0\} + S\{y_2 - y_0 + x_0 - x_1\} \\ &= L\{x_0 - x_1 + y_1 - y_0\} + S\{y_2 - y_1 + y_1 - y_0 + x_0 - x_1\} \\ &= L\{x_0 - x_1 + y_1 - y_0\} + S\{y_1 - y_0 + x_0 - x_1\} + S(y_2 - y_1) \end{aligned}$$

$$(1-S)\,(y_2 - y_1) \leq (L+S)\,\{y_1 - y_0 + x_0 - x_1\}$$

$$(y_2 - y_1) \leq \left(\frac{L+S}{1-S}\right)\{y_1 - y_0 + x_0 - x_1\}$$

$$\begin{aligned} y_1 - x_1 &= F(y_0, x_0) - F(x_0, y_0) \\ &\leq L\{y_0 - x_0 + y_0 - x_0\} + S\{F(y_0, x_0) - x_0 + y_0 - x_0\} \\ &\leq L\{y_0 - y_1 + y_0 - x_0\} + S\{y_1 - x_0 + y_0 - x_0\} \\ &\leq L\{y_0 - y_1 + y_1 - x_0 + y_0 - x_0\} + S\{y_1 - x_0 + y_0 - x_0\} \\ &\leq L\{y_0 - x_0 + y_0 - x_0\} + S\{y_1 - x_0 + y_0 - x_0\} \end{aligned}$$

Since $y_0 \geq y_1$

$$\begin{aligned} &\leq L\{y_0 - x_0 + y_0 - x_0\} + S\{y_0 - x_0 + y_0 - x_0\} \\ &\leq 2L\{y_0 - x_0\} + 2S\{y_0 - x_0\} \end{aligned}$$

$$y_1 - x_1 \leq 2(L+S)\{x_0 - y_0\}$$

Again

$$\begin{aligned} x_3 - x_2 &= F(x_2, y_2) - F(x_1, y_1) \\ &\leq L\{y_1 - y_2 + x_2 - x_1\} + S\{F(x_2, y_2) - x_1 + y_1 - y_2\} \\ &= L\{y_1 - y_2 + x_2 - x_1\} + S\{x_3 - x_1 + y_1 - y_2\} \\ &= L\{y_1 - y_2 + x_2 - x_1\} + S\{x_3 - x_1 + y_1 - y_2\} \\ &= L\{y_1 - y_2 + x_2 - x_1\} + S\{x_3 - x_2 + x_2 - x_1 + y_1 - y_2\} \\ &= L\{y_1 - y_2 + x_2 - x_1\} + S\{x_2 - x_1 + y_1 - y_2\} + S(x_3 - x_2) \end{aligned}$$

$$(1-S)\,(x_3 - x_2) \leq (L+S)\,\{y_1 - y_2 + x_2 - x_1\}$$

$$(x_3 - x_2) \leq \left(\frac{L+S}{1-S}\right)\{y_1 - y_2 + x_2 - x_1\}$$

from (3.17) and (3.18)

$$(x_3 - x_2) \leq \left(\frac{L+S}{1-S}\right)\left\{-\left(\frac{L+S}{1-S}\right)\{y_1 - y_0 + x_0 - x_1\} + \left(\frac{L+S}{1-S}\right)\{y_0 - y_1 + x_1 - x_0\}\right\}$$

$$(x_3 - x_2) \leq \left(\frac{L+S}{1-S}\right)^2 \{-\{y_1 - y_0 + x_0 - x_1\} + \{y_0 - y_1 + x_1 - x_0\}\}$$

$$(x_3 - x_2) \leq 2\left(\frac{L+S}{1-S}\right)^2 \{y_0 - y_1 + x_1 - x_0\} \quad \text{...(3.17)}$$

$$(x_3 - x_2) \leq \frac{1}{2}\left(\frac{2(L+S)}{1-S}\right)^2 \{y_0 - y_1 + x_1 - x_0\}$$

Similarly,

$$(y_3 - y_2) \leq \frac{1}{2}\left(\frac{2(L+S)}{1-S}\right)^2 \{y_0 - y_1 + x_1 - x_0\}$$

and
$y_2 - x_2 \leq \{2(L+S)\}^2\{y_0 - x_0\}$
Continue in this way, we get

$$(x_{n+1} - x_n) \leq \frac{1}{2}\left(\frac{2(L+S)}{1-S}\right)^n \{y_0 - y_1 + x_1 - x_0\}$$

$$(y_{n+1} - y_n) \leq \frac{1}{2}\left(\frac{2(L+S)}{1-S}\right)^n \{y_0 - y_1 + x_1 - x_0\}$$

$$(y_n - x_n) \leq \{2(L+S)\}^n\{y_0 - x_0\}$$ ...(3.20)

From the normality of K and (3.20)

$$\|x_{n+1} - x_n\| \leq N\left\|\frac{1}{2}\left(\frac{2(L+S)}{1-S}\right)^n\right\|\{\|x_0 - x_1 + y_0 - y_1\|\}$$

$$\|(y_{n+1} - y_n)\| \leq N\left\|\frac{1}{2}\left(\frac{2(L+S)}{1-S}\right)^n\right\|\{\|x_0 - x_1 + y_0 - y_1\|\}$$

$$\|y_n - x_n\| \leq N\|\{2(L+S)\}^n\|\,\|\{x_0 - y_0\}\| \ldots (3.21)$$

$$\|x_{n+1} - x_n\| \leq \frac{N}{2}\left\|\left(\frac{2(L+S)}{1-S}\right)^n\right\|\{\|x_0 - x_1 + y_0 - y_1\|\}$$

$$\|(y_{n+1} - y_n)\| \leq \frac{N}{2}\left\|\left(\frac{2(L+S)}{1-S}\right)^n\right\|\{\|x_0 - x_1 + y_0 - y_1\|\}$$

$$\|y_n - x_n\| \leq N\|\{2(L+S)\}^n\|\,\|\{x_0 - y_0\}\|$$

Since

$$\lim_{n\to\infty}\|(L+S)^n\| = r(L+S) < \frac{1}{2}$$

We have

$$\|\{2(L+S)\}^n\| \leq q^n \Rightarrow \frac{1}{2}\left\|\left(\frac{2(L+S)}{1-S}\right)^n\right\| \leq q^n$$ ... (3.22)

For some constant $q \in (0,1)$ and for sufficiently large n.
It follows from (3.21) and (3.22) that

$$\|x_{n+1} - x_n\| \leq Nq^n\|x_0 - x_1 + y_0 - y_1\|$$
$$\|y_{n+1} - y_n\| \leq Nq^n\|x_0 - x_1 + y_0 - y_1\|$$
$$\|y_n - x_n\| \leq Nq^n\|\{x_0 - y_0\}\|$$

Implies that $\{x_n\}$ and $\{y_n\}$ are two Cauchy sequences with same limit.
Let $\lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n = x^*$. Since K is closed, it is easy to know that
`

$$x_n \leq x^* \leq y_n \qquad \ldots(3.23)$$

For all $n \geq 0$
$F(x^*,x^*) - x_{n+1} = F(x^*,x^*) - F(x_n,y_n)$
By the normality of K we have

$$\|F(x^*,x^*) - x_{n+1}\| \leq N\|F(x^*,x^*) - F(x_n,y_n)\| \qquad \ldots(3.24)$$

This implies that

$$x^* = \lim_{n\to\infty} x_{n+1} = F(x^*,x^*)$$

This implies that $x^* \in M$ is a fixed point of F.
Now, we show that x^* is the unique fixed point of F.
Let $\bar{x} \in M$ be another fixed point of F. Since F is mixed increasing, we know that $x_n \leq x^* \leq y_n$ for all $n \geq 0$.
Since K is closed, it is easy to that $x^* \leq \bar{x} \leq x^*$. thus $x^* = \bar{x}$ this implies that $\bar{x}$ is the unique fixed point of F.

Application: Let E be a real Banach space induced by a closed convex normal cone P. " $\leq$ " be the partial ordering induced by P and N be the normal constant.
Let $C(I,E) = \{u: I \to E \text{ is continuous}\}$ and $P_c = \{u \in C(I,E): u(t) \geq 0,\ t \in I\}$ where $I = [0,1]$.For each $u \in C(I,E)$,
We define $\|u\|_c = \max_{t\in I}\|u(t)\|$.
Then $C(I,E)$ is a real Banach space with norm $\|.\|_c$ and P_c is closed convex normal cone with normal constant N. In this section, we also denote "$\leq$" by the partial ordering induced by P_c.
In the following, we consider the following Volterra type integral equation :

$$u(t) = x(t) + \int_0^t k(t,z)f(z,u(z),u(z))dz \ldots (4.1)$$

Where $x(t) \in C(I,E), f: I \times E \times E \to E$ and $k: I \times I: \to R$ is a nonnegative continuous function.

Theorem 4.1:- Let $u_0, v_0 \in C(I,E)$, $u_0 \le v_0$ and $D = \{u \in C(I,E): u_0 \le u \le v_0\}$. Suppose that the following conditions hold:

$C(1)$: $f(t,u(t),v(t))$ is measurable for any $u(t), v(t) \in C(I,E)$

$$C(2):\quad u_0(t) \le x(t) + \int_0^t k(t,z) f(z,u_0(z),v_0(s))dz$$

$$v_0(t) \le x(t) + \int_0^t k(t,z) f(z,v_0(z),u_0(s))dz$$

C(3): There exist two nonnegative constants L' and S' such that $u_1, u_2, v_1, v_2 \in C(I,E)$, $u_1 \le u_2$ and $v_2 \le v_1$ imply that

$$0 \le f(t,u_2,v_2) - f(t,u_1,v_1) \le L'\{v_1 - u_2 - u_1\}$$

$$+S'\{f(u_2,v_2) - u_1 + v_1 - v_2\}, \quad t \in I$$

C(4): There exist a constant $K \ge 0$ such that,

$$\int_I k(t,z)\,dz \le K \; for\ each\ t \in I$$

C(5): $r(L' + S') < \frac{1}{2}$

$$\text{Let } u_n(t) \le x(t) + \int_0^t k(t,z) f(z,u_{n-1}(z),v_{n-1}(z))dz$$

$n = 1,2,3,\ldots$

….(4.2)

$$v_n(t) \le x(t) + \int_0^t k(t,z) f(z,v_{n-1}(z),u_{n-1}(z))dz$$

Then $\{u_n(t)\}$ and $\{v_n(t)\}$ both converges uniformly to the unique solution $w(t)$ of (4.1)

Proof:- Define $A: D \times D \to C(I,E)$ as follows:

$$A(u,v) = x(t) + \int_0^t k(t,z) f(z,u(z),v(z))dz \quad ….(4.3)$$

From C(2) and (4.2), we know that $u_0 \le A(u_0, v_0)$ and $A(v_0, u_0) \le v_0$, since k(t,s) is nonnegative and continuous, it follows from C(3) that

$$0 \le A(u_2,v_2) - A(u_1,v_1) = \int_0^t k(t,z)\big(f(z,u_2(z),v_2(z)) - f(z,u_1(z),v_1(z))\big)dz$$

$$0 \le A(u_2,v_2) - A(u_1,v_1)$$

$$\le \int_0^t k(t,z) L'\{v_1(z) + u_2(z) - u_1(z)\} + S'\{f(u_2(z),v_2(z)) - u_1(z) + v_1(z) - v_2(z)\}\,dz$$

$$\le \int_0^t k(t,z) L'\{v_1(z) + u_2(z) - u_1(z)\}\,dz + \int_0^t k(t,z)\, S'\{f(u_2(z),v_2(z)) - u_1(z) + v_1(z) - v_2(z)\}\,dz$$

$$\le L'\{v_1(z) + u_2(z) - u_1(z)\} + S'\{f(u_2(z),v_2(z)) - u_1(z) + v_1(z) - v_2(z)\}$$

Where,

$$Lu(t) = \int_0^t L' k(t,z) u(z) dz \quad \text{And} \quad Su(t) = \int_0^t S' k(t,z) u(z) dz$$

It follows from C(4) and C(5) that L , S are two positive linear operators (in the sense that L is positive if $Lu \ge 0$ whenever $u \ge 0$) with $r(L+S) < \frac{1}{2}$

By theorem 3.2, we know that A admits a unique fixed point $w(t) \in C(I,E)$.

Further, from the proof of theorem 3.2, we know that $u_n(t)$ and $v_n(t)$ both converges uniformly to the unique solution of (4.1). The proof is complete.

References

[1]. H.Amann, Fixed Point Equations and Nonlinear Eigen value Problems in Ordered Banach Space SIAM Rev. 18 (1976) 620-709.

[2]. D.J.Guo and V. Lakshmikantham, Coupled fixed points of nonlinear

operators with applications, Nonlinear Anal.TMA 11 (1987) 623-632.
[3]. Shihsen Chang and Yihai Ma, Coupled fixed points for mixed monotone condensing operators and an existence theorem of the solutions for a class of functional equations, J.Math. Anal. Appl. **160** (1991), 468479.
[4]. Yongzhuo Chen, Existence theorems of coupled fixed points, J. Math. Anal. Appl. **154** (1991), 142150.
[5]. K. Deimling, Nonlinear Functional Analysis, SpringerVerlag, 1985.
[6]. D.J. Guo, Fixed points of mixed monotone operators with applications Appl. Anal. 31 (1988) 215-24.
[7]. S.S Chang. Y.J. Cho and N.J. Huang,Coupled Fixed Point Theorems With Applications Korean Math. Soc. 33(3) (1996) 575-85.
[8]. S.Heikila and V. Lakshmikantham, nonlinear Diff. Eqs. Dekker, New York, 1994.
[9]. M.A. Krasnoselskii and P.P. Zabriko, Geometrical Methods of Nonlinear Analysis, Springer-Verlag, Berlin, 1984.
[10]. B.Y. Nguyen and H.K. Nguyen, J. math. Anal. Appl. 250 (2000) 368-71.