CONDITONAL DEFORMATION RETRACT OF TIMELIKE HELIX IN MINKOWSKI 3-SPACE

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ABSTRACT: Our aim in the present article is to introduce and study new types of retraction of time like helix in Minkowski 3-space. Types of the deformation retracts of the time like helix in Minkowski 3-space are presented. The relations between the retraction and the deformation retract of time like helix are deduced. Types of minimal retraction of time like helix in Minkowski 3-space are also presented. New types of homotopy maps are described.

Keywords. Minkowski 3-space, time like helix, retractions, deformation retracts.


INTRODUCTION AND DEFINITIONS

As is well known, the theory of deformation retractions is always one of interesting topics in Euclidian and Non-Euclidian space and it has been investigated from the various viewpoints by many branches of topology and differential geometry (El-Ahmady A. E. 2011 a, b, c), (El-Ahmady, A. E. 2007 b), (El-Ahmady, A. E. 2006).

An n-dimensional topological manifold M is a Hausdorff topological space with a countable basis for the topology which is locally homeomorphic to \( \mathbb{R}^n \). If \( h: U \rightarrow U' \) is a homeomorphism of \( U \subseteq M \) onto \( U' \subseteq \mathbb{R}^n \), then \( h \) is called a chart of \( M \) and \( U \) is the associated chart domain. A collection \( \{ U_a : a \in \Lambda \} \) is said to be an atlas for \( M \) if \( \bigcup_{a \in \Lambda} U_a = M \). Given two charts \( (U_a, h_a) \) and \( (U_b, h_b) \) such that \( U_{ab} = U_a \cap U_b \neq \emptyset \), the transformation chart \( h_{ab} = h_b \circ h_a^{-1} \) between open sets of \( \mathbb{R}^n \) is defined and if all of these charts transformation are \( C^\infty \)-mappings, then the manifolds under consideration is a \( C^\infty \)-
A differentiable structure on $M$ is a differentiable atlas and a differentiable manifolds is a topological manifolds with a differentiable structure (Arkowitz, M. 2011), (Reid M, Balazs S. 2005), (Shick Pl. 2007) and (Strom, J. 2011).

A subset $A$ of a topological space $X$ is called a retract of $X$ if there exists a continuous map $r : X \rightarrow A$ such that $(r \circ i) = a$ for $a \in A$, where $A$ is closed and $X$ is open (El-Ahmady, A. E. 2012a, b, c).

This can be restated as follows. If $i : A \rightarrow X$ is the inclusion map, then $r : X \rightarrow A$ is a map such that $r \circ i = \text{id}_A$.

Another simple-but extremely useful-idea is that of a retract. If $A X \subseteq M$, then $A$ is a retract of $X$ if there is a commutative diagram (El-Ahmady, A. E. 2011a).
A subset $A$ of a topological space $X$ is a deformation retract of $X$ if there exists a retraction $r: X \to A$ and a homotopy $\varphi_t: X \times I \to X$ such that (Naber, G. L. 2011)

\[
\begin{align*}
\varphi(x, 0) &= x \\
\varphi(x, 1) &= r(x) \\
\varphi(a, t) &= a, \quad a \in A, \quad t = [0,1].
\end{align*}
\]

Also, suppose $X$ is a topological space, $A$ is a subspace of $X$, and $r: X \to A$ is a retraction. We say that $r$ is a deformation retract if $\varphi_t \circ r$ is homotopic to the identity map of $X$, where $\varphi_t: A \to X$ is the inclusion map. If there exists a deformation retract from $X$ to $A$, then $A$ is said to be a deformation retract of $X$. Because $\varphi_t \circ r = I_X$ and $r \circ \varphi_t = I_A$, it follows that both $\varphi_t$ and $r$ are homotopy equivalence (Strom, J. 2011).

Let $M^{n+1}$ be the Minkowski $(n+1)$-space, that is, $\mathbb{R}^{n+1}$ endowed with the standard flat metric $ds^2 = \Sigma_{i=1}^{n+1} dx_i^2$. Also, $ds^2 > 0$, $ds^2 = 0$ and $ds^2 < 0$ correspond to space–like, null and time–like geodesics (James, B. H. 2003).

Helix is one of the most fascinating curves in Science and Nature. From the view of differential geometry, a helix is a geometry curve with non–vanishing constant curvature (or first curvature of the curve and denoted by $k_1$) and non-vanishing constant torsion (or second curvature of the curve and denoted by $k_2$).

Indeed a helix is a special case of the general helix. A curve of constant slope or general helix in Euclidean 3-space $\mathbb{R}^3$ is defined by the property that the tangent makes a constant angle with a fixed straight line (Yaliniz, A. F. 2007). It is well known that to each unit speed curve $\alpha: I \subset \mathbb{R} \to \mathbb{R}^3$ in the Euclidean space $\mathbb{R}^3$ whose successive derivatives $\alpha'(s)$, $\alpha''(s)$ and $\alpha''''(s)$ are linearly independent vectors, one can associate three mutually orthogonal unit vector field T,N and B called respectively the tangent, the principal normal and the binormal vector field Ceylan. C. A. (1 (2007). At each point $\alpha(s)$ of curve $\alpha$, the planes spanned by $\{T,N\}$ and $\{N,B\}$ are known respectively as the osculating plane, the rectifying plane and the normal plane (Kocayigit, H. 2007). The curves $\alpha: I \subset \mathbb{R} \to \mathbb{R}^3$ for which the position vector $\alpha$ always lie in their rectifying plane are simplicity called rectifying curves. Similarly, the curves for which the position vector $\alpha$ always lie in their osculating plane, are for simplicity called osculating curves. Moreover, the curves for which the position vector $\alpha$ always lie in their normal plane, are for simplicity called normal curves (Ilarslan, K. 2004).

The Minkowski 3-space $\mathbb{E}^3$ is the Euclidean 3-space $\mathbb{R}^3$ provided with the standard flat metric given by $g = -dx_1^2 + dx_2^2 + dx_3^2$.

Where $(x_1, x_2, x_3)$ is a rectangular coordinate system of $\mathbb{E}^3$. Since $g$ is an indefinite metric, recall that a vector $v \in \mathbb{E}^3$ can have one of three Lorentzian causal characters, it can be space like if $g(v, v) > 0$ or $v = 0$, time like if $g(v, v) < 0$ and light like if $g(v, v) = 0$ and $v = 0$. Similarly, an arbitrary curve $\alpha = (x(t))$ in $\mathbb{E}^3$ can locally be space like, time like or light like, if all of its velocity vectors $\alpha'(t)$ are respectively space.
like time like or light like respectively (Onder, M., 2006).

Minkowski space is originally from the relativity in physics. In fact, a time like curve corresponds to the path of an observer moving at less than the speed of light, a light like curves correspond to moving at the speed of light and a space like curves moving faster than light (El-Ahmady, A. E. 2013 a, b, c, d).

Consider the moving Frenet frame \{T, N, B\} along the curve \(\alpha(s)\) in \(E^3\). For an arbitrary curve \(\alpha(s)\) in the space \(E^3\), the following Frenet formula are given in (El-Ahmady, A. E. 2013 a, b, c, d).

A curve in Lorentzian space \(L^n\) is a smooth map \(\alpha: I \rightarrow L^n\) where \(I\) is the open interval in the real line \(R\). The interval \(I\) has a coordinate system consisting of the identity map \(u\) of \(I\). The velocity of \(\alpha\) at \(t \in I\) is \(\alpha'(t) = \frac{d\alpha}{ds}\). A curve \(\alpha\) is said to be regular if \(\alpha'(t)\) does not vanish for all \(t \in I\). \(\alpha(s)\) is space like if its velocity vectors \(\alpha'(t)\) are space like for all \(t \in I\), similarly for time like and null. If \(\alpha\) is a null curve, we can reparametrize it such that \(u''(0) \cdot u'(0) = 0\) and \(u'(0) \neq 0\).

If \(\alpha\) is null curve, then the Frenet formula read

\[
\left(\begin{array}{c}
\alpha'(s) \\
\alpha''(s)
\end{array}\right) = \left(\begin{array}{cc}
u'(s)&v''(s) \\
\nu(s)&v'(s)
\end{array}\right)
\]

Where

\[
\begin{align*}
g(T,T) &= g(T,B) = 0, g(N,N) = 1, \\
g(T,N) &= g(N,B) = 0 \text{ and } g(T,B) = 1.
\end{align*}
\]

In this case, \(K_4\) can take only two values \(K_4 = 0\) when \(\alpha\) is straight line or \(K_4 = 1\) in all other cases (El-Ahmady, A. E. 2013 a, b, c, d, e, f).

The importance of the study of null curve and its presence in the physical theories is clear from the fact that the classical relativistic string is a surface or world sheet in Minkowski space which satisfies the Lorentzian analogue of minimal surface equation. The string equations simplify to the wave equation and a couple of extra simple equations, and by solving the 2-dimentional wave equations it turns out that string are equivalent to pairs of null curves, or a single null curve in the case of an open string. In this paper we will introduce a new type of deformation retractions of time like curve in Minkowski space from view point of the geodesic deformation retract of Klein bottle and its folding and retrraction of chaotic black hole, as presented by (El-Ahmady, A. E. 2011b) and (El-Ahmady, A. E. 2011c).

**THE MAIN RESULTS**

Let \(\alpha(s)\) be a time like helix. Then we can write its position vector as follows:

\[
\alpha(s) = \lambda(s)T(s) + \mu(s)N(s) + \nu(s)B(s)
\]

Differentiating equation (1) with respect to \(s\) and using Frenet formula we have

\[
\lambda'(s) + \mu(s)K_4 = 0, \lambda'(s) + \mu'(s) - \nu(s)K_3 = 0, \mu'(s)K_2 + \nu'(s) = 0
\]

From equation (2) we have
\[ \mu'' - \mu(\omega) (K_1^2 - K_2^2) + K_2 = 0 \]  

(3)

Now we are going to discuss the following case

If \( K_1^2 < K_2^2 \). The solution of equation (3) is

\[ \mu(\omega) = C_1 \cos(S \sqrt{K_1^2 - K_2^2}) + C_2 \sin(S \sqrt{K_1^2 - K_2^2}) - \frac{K_2^2}{K_1^2 - K_2^2} \]  

(4)

Where \( C_1, C_2 \in \mathbb{R} \). From equation (2) we have \( \lambda(\omega) = 1 - \mu(\omega)K_2 \). By using (4) we find the solution of this equation as follows:

\[ h(\omega) = S - \frac{C_1}{\sqrt{K_1^2 - K_2^2}} \sin(S \sqrt{K_1^2 - K_2^2}) + \frac{C_2}{\sqrt{K_1^2 - K_2^2}} \cos(S \sqrt{K_1^2 - K_2^2}) + \frac{K_2^2}{K_1^2 - K_2^2} T(\omega) + \]

\[ + (C_1 \cos(S \sqrt{K_1^2 - K_2^2}) + C_2 \sin(S \sqrt{K_1^2 - K_2^2}) - \frac{K_2^2}{K_1^2 - K_2^2}) N(\omega) + \]

\[ - \frac{C_1}{\sqrt{K_1^2 - K_2^2}} \sin(S \sqrt{K_1^2 - K_2^2}) + \frac{C_2}{\sqrt{K_1^2 - K_2^2}} \cos(S \sqrt{K_1^2 - K_2^2}) + \frac{K_2^2}{K_1^2 - K_2^2} B(\omega). \]

(5)

Also from (2) we have \( v(\omega) = -\mu(\omega)K_2 \). By using (4) we get

\[ v(\omega) = -\frac{C_1}{\sqrt{K_1^2 - K_2^2}} \sin(S \sqrt{K_1^2 - K_2^2}) + \frac{C_2}{\sqrt{K_1^2 - K_2^2}} \cos(S \sqrt{K_1^2 - K_2^2}) + \frac{K_2^2}{K_1^2 - K_2^2}. \]

(6)

Hence, the position vector \( a_2(\omega) \) is

\[ a_2(\omega) = (S - \frac{C_1}{\sqrt{K_1^2 - K_2^2}} \sin(S \sqrt{K_1^2 - K_2^2}) + \frac{C_2}{\sqrt{K_1^2 - K_2^2}} \cos(S \sqrt{K_1^2 - K_2^2}) + \frac{K_2^2}{K_1^2 - K_2^2}) T(\omega) + \]

\[ + (C_1 \cos(S \sqrt{K_1^2 - K_2^2}) + C_2 \sin(S \sqrt{K_1^2 - K_2^2}) - \frac{K_2^2}{K_1^2 - K_2^2}) N(\omega) + \]

\[ - \frac{C_1}{\sqrt{K_1^2 - K_2^2}} \sin(S \sqrt{K_1^2 - K_2^2}) + \frac{C_2}{\sqrt{K_1^2 - K_2^2}} \cos(S \sqrt{K_1^2 - K_2^2}) + \frac{K_2^2}{K_1^2 - K_2^2} B(\omega). \]

Now we are going to discuss the retraction of the position vector \( a_2(\omega) \) as follows

Let \[ \eta_2(\omega) = a_2(\omega) - \beta \], where \( a_2(\omega) - \beta \) be the open helix and \( a_2(\omega) - \beta^* \) be the retraction of the position vector \( a_2(\omega) \).

Now we discuss the following cases:

If \( C_1 = 0 \), we have the following retraction defined as:

\[ \eta_2(a_2(\omega)) = (S - \frac{C_2}{\sqrt{K_1^2 - K_2^2}} \sin(S \sqrt{K_1^2 - K_2^2}) + \frac{K_2^2}{K_1^2 - K_2^2}) T(\omega) + \]

\[ + (C_2 \sin(S \sqrt{K_1^2 - K_2^2}) - \frac{K_2^2}{K_1^2 - K_2^2}) N(\omega) + (\frac{C_2}{\sqrt{K_1^2 - K_2^2}} \cos(S \sqrt{K_1^2 - K_2^2}) + \frac{K_2^2}{K_1^2 - K_2^2}) B(\omega). \]

Also ,if \( C_2 = 0 \) we obtain the following retraction defined by:
If $K_E = 0$, we present the following retraction given by

$$\eta_3(\alpha_2(z)) = (S - \frac{G_{K_E}}{\sqrt{r^2 - R^2}} \sin(S\sqrt{R^2 - K^2_E}) + \frac{K_{E}^2}{r^2 - K^2_E}) T(z)$$
$$+(C_L \cos(S\sqrt{R^2 - K^2_E}) - \frac{K_{E}^2}{r^2 - K^2_E}) N(z) + \left(-\frac{G_{K_E}}{\sqrt{r^2 - R^2}} \sin(S\sqrt{R^2 - K^2_E}) + \frac{K_{E}^2}{r^2 - K^2_E}\right) B(z) +$$

$$\left(C_L \cos(S\sqrt{R^2 - K^2_E}) + C_L \sin(S\sqrt{R^2 - K^2_E})\right) N(z).$$

If $T(z) = 0$, we present the following retraction given by:

$$\eta_3(\alpha_2(z)) = \left(S - \frac{G_{K_E}}{\sqrt{r^2 - R^2}} \sin(S\sqrt{R^2 - K^2_E}) + \frac{K_{E}^2}{r^2 - K^2_E}\right) T(z) + \left(C_L \cos(S\sqrt{R^2 - K^2_E}) + C_L \sin(S\sqrt{R^2 - K^2_E}) + \frac{K_{E}^2}{r^2 - K^2_E}\right) N(z).$$

Also, if $N(z) = 0$ we obtain the following retraction defined as:

$$\eta_3(\alpha_2(z)) = \left(S - \frac{G_{K_E}}{\sqrt{r^2 - R^2}} \sin(S\sqrt{R^2 - K^2_E}) + \frac{K_{E}^2}{r^2 - K^2_E}\right) T(z) + \left(-\frac{G_{K_E}}{\sqrt{r^2 - R^2}} \sin(S\sqrt{R^2 - K^2_E}) + \frac{K_{E}^2}{r^2 - K^2_E}\right) B(z).$$

Now, if $B(z) = 0$ we present the following retraction given by:

$$\eta_3(\alpha_2(z)) = \left(S - \frac{G_{K_E}}{\sqrt{r^2 - R^2}} \sin(S\sqrt{R^2 - K^2_E}) + \frac{K_{E}^2}{r^2 - K^2_E}\right) T(z) + \left(C_L \cos(S\sqrt{R^2 - K^2_E}) + C_L \sin(S\sqrt{R^2 - K^2_E}) + \frac{K_{E}^2}{r^2 - K^2_E}\right) N(z).$$

Also, if $T(z) = N(z) = 0$, we have the following retraction defined as:

$$\eta_3(\alpha_2(z)) = \left(-\frac{G_{K_E}}{\sqrt{r^2 - R^2}} \sin(S\sqrt{R^2 - K^2_E}) + \frac{K_{E}^2}{r^2 - K^2_E}\right) B(z).$$

Now, if $T(z) = B(z) = 0$ we present the following retraction given by:

$$\eta_3(\alpha_2(z)) = \left(C_L \cos(S\sqrt{R^2 - K^2_E}) + C_L \sin(S\sqrt{R^2 - K^2_E}) + \frac{K_{E}^2}{r^2 - K^2_E}\right) N(z).$$

Also, if $N(z) = B(z) = 0$, we have the following retraction defined as:
\[ \eta_{14}(\alpha_2(\beta)) = (S - \frac{\hat{c}_2}{\sqrt{R_2^2 - R_1^2}} \sin(\sqrt{R_2^2 - R_1^2}) + \frac{\hat{c}_2}{\sqrt{R_2^2 - R_1^2}} \cos(\sqrt{R_2^2 - R_1^2}) + \frac{\hat{c}_2}{R_2^2 - R_1^2})^T(\beta). \]

Moreover, if \( T(\beta) = N(\beta) = B(\beta) = 0 \), we have the following minimal retraction defined as:

\[ \eta_{14}(\alpha_2(\beta)) = (0, 0, 0). \]

In this position, we present some cases of deformation retract of time like helix in Minkowski 3-space. The deformation retract of time like helix is

\[ \varphi: [\alpha_2(\alpha) - \beta] \times 1 \rightarrow [\alpha_2(\alpha) - \beta], \]

where \([\alpha_2(\alpha) - \beta]\) be the open helix of the position vector \(\alpha_2(\alpha)\) and 1 is the closed interval \([0, 1]\). be present as

\[ \varphi(\chi, h) = (((S - \frac{\hat{c}_2}{\sqrt{R_2^2 - R_1^2}} \sin(\sqrt{R_2^2 - R_1^2}) + \frac{\hat{c}_2}{\sqrt{R_2^2 - R_1^2}} \cos(\sqrt{R_2^2 - R_1^2}) + \frac{\hat{c}_2}{R_2^2 - R_1^2})^T(\beta) + (C_1 \cos(\sqrt{R_2^2 - R_1^2}) + C_2 \sin(\sqrt{R_2^2 - R_1^2}) - \frac{\hat{c}_2}{R_2^2 - R_1^2})^T(\beta) + (\frac{\hat{c}_2}{\sqrt{R_2^2 - R_1^2}} \sin(\sqrt{R_2^2 - R_1^2}) + \frac{\hat{c}_2}{\sqrt{R_2^2 - R_1^2}} \cos(\sqrt{R_2^2 - R_1^2}) - \frac{\hat{c}_2}{R_2^2 - R_1^2})^T(\beta) - \beta) \times 1 \]

\[ = \left( (S - \frac{\hat{c}_2}{\sqrt{R_2^2 - R_1^2}} \sin(\sqrt{R_2^2 - R_1^2}) + \frac{\hat{c}_2}{\sqrt{R_2^2 - R_1^2}} \cos(\sqrt{R_2^2 - R_1^2}) + \frac{\hat{c}_2}{R_2^2 - R_1^2})^T(\beta) + (C_1 \cos(\sqrt{R_2^2 - R_1^2}) + C_2 \sin(\sqrt{R_2^2 - R_1^2}) - \frac{\hat{c}_2}{R_2^2 - R_1^2})^T(\beta) + (\frac{\hat{c}_2}{\sqrt{R_2^2 - R_1^2}} \sin(\sqrt{R_2^2 - R_1^2}) + \frac{\hat{c}_2}{\sqrt{R_2^2 - R_1^2}} \cos(\sqrt{R_2^2 - R_1^2}) - \frac{\hat{c}_2}{R_2^2 - R_1^2})^T(\beta) - \beta) \times 1 \]

The deformation retract of the helix \(\alpha_2(\alpha)\) into the retraction \(\eta_{14}(\alpha_2(\alpha))\) is

\[ \varphi(\alpha, h) = (1 - h) \left( (S - \frac{\hat{c}_2}{\sqrt{R_2^2 - R_1^2}} \sin(\sqrt{R_2^2 - R_1^2}) + \frac{\hat{c}_2}{\sqrt{R_2^2 - R_1^2}} \cos(\sqrt{R_2^2 - R_1^2}) + \frac{\hat{c}_2}{R_2^2 - R_1^2})^T(\beta) + (C_1 \cos(\sqrt{R_2^2 - R_1^2}) + C_2 \sin(\sqrt{R_2^2 - R_1^2}) - \frac{\hat{c}_2}{R_2^2 - R_1^2})^T(\beta) + (\frac{\hat{c}_2}{\sqrt{R_2^2 - R_1^2}} \sin(\sqrt{R_2^2 - R_1^2}) + \frac{\hat{c}_2}{\sqrt{R_2^2 - R_1^2}} \cos(\sqrt{R_2^2 - R_1^2}) - \frac{\hat{c}_2}{R_2^2 - R_1^2})^T(\beta) - \beta) \times 1 \right) + h \left( (S + \frac{\hat{c}_2}{\sqrt{R_2^2 - R_1^2}} \sin(\sqrt{R_2^2 - R_1^2}) + \frac{\hat{c}_2}{\sqrt{R_2^2 - R_1^2}} \cos(\sqrt{R_2^2 - R_1^2}) + \frac{\hat{c}_2}{R_2^2 - R_1^2})^T(\beta) - \beta) \times 1 \right). \]
Where
\[
\varphi(x, 0) = \left(\left( S - \frac{S \alpha_k}{\sqrt{R^2 - K^2}} \sin(S \sqrt{R^2 - K^2}) + \frac{S \alpha_k}{\sqrt{R^2 - K^2}} \cos(S \sqrt{R^2 - K^2}) + \frac{K \beta}{\sqrt{R^2 - K^2}} \right) T(x) + (C \cos(S \sqrt{R^2 - K^2}) + C \sin(S \sqrt{R^2 - K^2}) - \frac{\beta}{\sqrt{R^2 - K^2}}) N(x) + \left( -\frac{S \alpha_k}{\sqrt{R^2 - K^2}} \right) B(x) \right) - \beta, \right) + \left( C \cos(S \sqrt{R^2 - K^2}) - \frac{\beta}{\sqrt{R^2 - K^2}} \right) N(x) + \left( -\frac{S \alpha_k}{\sqrt{R^2 - K^2}} \sin(S \sqrt{R^2 - K^2}) + \frac{K \beta}{\sqrt{R^2 - K^2}} \right) B(x)),
\]
and
\[
\varphi(x, 1) = \left(\left( S - \frac{S \alpha_k}{\sqrt{R^2 - K^2}} \cos(S \sqrt{R^2 - K^2}) + \frac{S \alpha_k}{\sqrt{R^2 - K^2}} \sin(S \sqrt{R^2 - K^2}) + \frac{K \beta}{\sqrt{R^2 - K^2}} \right) T(x) + (C \cos(S \sqrt{R^2 - K^2}) + C \sin(S \sqrt{R^2 - K^2}) - \frac{\beta}{\sqrt{R^2 - K^2}}) N(x) + \left( -\frac{S \alpha_k}{\sqrt{R^2 - K^2}} \cos(S \sqrt{R^2 - K^2}) + \frac{K \beta}{\sqrt{R^2 - K^2}} \right) B(x) \right) - \beta, \right) + \left( C \cos(S \sqrt{R^2 - K^2}) - \frac{\beta}{\sqrt{R^2 - K^2}} \right) N(x) + \left( -\frac{K \beta}{\sqrt{R^2 - K^2}} \right) B(x)),
\]

The deformation retract of the helix \( \alpha_2(s) \) into the retraction \( \eta_2(\alpha_2(s)) \) is defined as
\[
\varphi(x, h) = \cos \frac{h}{N} \left(\left( S - \frac{S \alpha_k}{\sqrt{R^2 - K^2}} \sin(S \sqrt{R^2 - K^2}) + \frac{S \alpha_k}{\sqrt{R^2 - K^2}} \cos(S \sqrt{R^2 - K^2}) + \frac{K \beta}{\sqrt{R^2 - K^2}} \right) T(x) + (C \cos(S \sqrt{R^2 - K^2}) + C \sin(S \sqrt{R^2 - K^2}) - \frac{\beta}{\sqrt{R^2 - K^2}}) N(x) + \left( -\frac{S \alpha_k}{\sqrt{R^2 - K^2}} \sin(S \sqrt{R^2 - K^2}) + \frac{K \beta}{\sqrt{R^2 - K^2}} \right) B(x) \right) - \beta, \right) + \left( C \cos(S \sqrt{R^2 - K^2}) - \frac{\beta}{\sqrt{R^2 - K^2}} \right) N(x) + \left( -\frac{S \alpha_k}{\sqrt{R^2 - K^2}} \sin(S \sqrt{R^2 - K^2}) + \frac{K \beta}{\sqrt{R^2 - K^2}} \right) B(x)),
\]
and
\[
\varphi(x, h) = \frac{1 - h}{1 + h} \left(\left( S - \frac{S \alpha_k}{\sqrt{R^2 - K^2}} \sin(S \sqrt{R^2 - K^2}) + \frac{S \alpha_k}{\sqrt{R^2 - K^2}} \cos(S \sqrt{R^2 - K^2}) + \frac{K \beta}{\sqrt{R^2 - K^2}} \right) T(x) + (C \cos(S \sqrt{R^2 - K^2}) + C \sin(S \sqrt{R^2 - K^2}) - \frac{\beta}{\sqrt{R^2 - K^2}}) N(x) + \left( -\frac{S \alpha_k}{\sqrt{R^2 - K^2}} \sin(S \sqrt{R^2 - K^2}) + \frac{K \beta}{\sqrt{R^2 - K^2}} \right) B(x) \right) - \beta, \right) + \left( C \cos(S \sqrt{R^2 - K^2}) - \frac{\beta}{\sqrt{R^2 - K^2}} \right) N(x) + \left( -\frac{S \alpha_k}{\sqrt{R^2 - K^2}} \sin(S \sqrt{R^2 - K^2}) + \frac{K \beta}{\sqrt{R^2 - K^2}} \right) B(x)),
\]

The deformation retract of the helix \( \alpha_2(s) \) into the retraction \( \eta_3(\alpha_2(s)) \) is given by
\[
\varphi(x, h) = \left(\left( S - \frac{S \alpha_k}{\sqrt{R^2 - K^2}} \sin(S \sqrt{R^2 - K^2}) + \frac{S \alpha_k}{\sqrt{R^2 - K^2}} \cos(S \sqrt{R^2 - K^2}) + \frac{K \beta}{\sqrt{R^2 - K^2}} \right) T(x) + (C \cos(S \sqrt{R^2 - K^2}) + C \sin(S \sqrt{R^2 - K^2}) - \frac{\beta}{\sqrt{R^2 - K^2}}) N(x) + \left( -\frac{S \alpha_k}{\sqrt{R^2 - K^2}} \sin(S \sqrt{R^2 - K^2}) + \frac{K \beta}{\sqrt{R^2 - K^2}} \right) B(x) \right) - \beta, \right) + \left( C \cos(S \sqrt{R^2 - K^2}) - \frac{\beta}{\sqrt{R^2 - K^2}} \right) N(x) + \left( -\frac{S \alpha_k}{\sqrt{R^2 - K^2}} \sin(S \sqrt{R^2 - K^2}) + \frac{K \beta}{\sqrt{R^2 - K^2}} \right) B(x)),
\]
is defined as

\[
\phi(x, h) = e^{\beta (S - \frac{G_K}{r^2 - 2})} \left( C_1 \cos \left( \sqrt{r^2 - 2} - K_2^2 \right) + \frac{G_K}{r^2 - 2} \right) \left( \cos \left( \sqrt{r^2 - 2} - K_2^2 \right) + \frac{G_K}{r^2 - 2} \right) T(x)
\]

\[+(C_1 \cos \left( \sqrt{r^2 - 2} - K_2^2 \right) + C_2 \sin \left( \sqrt{r^2 - 2} - K_2^2 \right) - \frac{G_K}{r^2 - 2}) N(x) + \]

\[
- \frac{G_K}{r^2 - 2} \left( \sin \left( \sqrt{r^2 - 2} - K_2^2 \right) + \frac{G_K}{r^2 - 2} \right) \left( \cos \left( \sqrt{r^2 - 2} - K_2^2 \right) + \frac{G_K}{r^2 - 2} \right) B(x)
\]

The deformation retract of the helix \( u_2(\omega) \) into the retraction \( \eta_2(\omega) \) is defined as

\[
\phi(x, h) = e^{(1 - h)(S - \frac{G_K}{r^2 - 2})} \left( C_1 \cos \left( \sqrt{r^2 - 2} - K_2^2 \right) + \frac{G_K}{r^2 - 2} \right) \left( \cos \left( \sqrt{r^2 - 2} - K_2^2 \right) + \frac{G_K}{r^2 - 2} \right) T(x)
\]

\[+(C_1 \cos \left( \sqrt{r^2 - 2} - K_2^2 \right) + C_2 \sin \left( \sqrt{r^2 - 2} - K_2^2 \right) - \frac{G_K}{r^2 - 2}) N(x) + \]

\[
- \frac{G_K}{r^2 - 2} \left( \sin \left( \sqrt{r^2 - 2} - K_2^2 \right) + \frac{G_K}{r^2 - 2} \right) \left( \cos \left( \sqrt{r^2 - 2} - K_2^2 \right) + \frac{G_K}{r^2 - 2} \right) B(x)
\]
The deformation retract of the helix $\alpha_2(t)$ into the retraction $\eta_0(\alpha_2(t))$ is defined by

$$\varphi(x, h) = \frac{1}{2} \left( 2h + \frac{1}{2} \right) \left( S - \frac{S R^2}{R^2 - R_1^2} \sin\left( S \sqrt{R^2 - R_1^2} \right) + \frac{S R^2}{R^2 - R_1^2} \cos\left( S \sqrt{R^2 - R_1^2} \right) + \frac{R^2}{R_1^2 - R_2^2} T(x) + 2n \cos\left( S \sqrt{R^2 - R_1^2} \right) \right) N(x) +$$

$$+ \left( \frac{S R^2}{R^2 - R_1^2} \sin\left( S \sqrt{R^2 - R_1^2} \right) + \frac{S R^2}{R^2 - R_1^2} \cos\left( S \sqrt{R^2 - R_1^2} \right) + \frac{R^2}{R_1^2 - R_2^2} B(x) \right) - \frac{R_2}{R_1^2 - R_2^2} N(x).$$

The deformation retract of the helix $\alpha_2(t)$ into the retraction $\eta_1(\alpha_2(t))$ is defined as

$$\varphi(x, h) = \text{Int}^{(t\rightarrow\infty)} \left( S - \frac{S R^2}{R^2 - R_1^2} \sin\left( S \sqrt{R^2 - R_1^2} \right) + \frac{S R^2}{R^2 - R_1^2} \cos\left( S \sqrt{R^2 - R_1^2} \right) + \frac{R^2}{R_1^2 - R_2^2} T(x) + 2n \cos\left( S \sqrt{R^2 - R_1^2} \right) \right) N(x) +$$

$$+ \left( \frac{S R^2}{R^2 - R_1^2} \sin\left( S \sqrt{R^2 - R_1^2} \right) + \frac{S R^2}{R^2 - R_1^2} \cos\left( S \sqrt{R^2 - R_1^2} \right) + \frac{R^2}{R_1^2 - R_2^2} B(x) \right) - \frac{R_2}{R_1^2 - R_2^2} N(x).$$

The deformation retract of the helix $\alpha_2(t)$ into the retraction $\eta_{10}(\alpha_2(t))$ is defined as

$$\varphi(x, h) = e^h \left( 1 - h \right) \left( S - \frac{S R^2}{R^2 - R_1^2} \sin\left( S \sqrt{R^2 - R_1^2} \right) + \frac{S R^2}{R^2 - R_1^2} \cos\left( S \sqrt{R^2 - R_1^2} \right) + \frac{R^2}{R_1^2 - R_2^2} T(x) + 2n \cos\left( S \sqrt{R^2 - R_1^2} \right) \right) N(x) +$$

$$+ \left( \frac{S R^2}{R^2 - R_1^2} \sin\left( S \sqrt{R^2 - R_1^2} \right) + \frac{S R^2}{R^2 - R_1^2} \cos\left( S \sqrt{R^2 - R_1^2} \right) + \frac{R^2}{R_1^2 - R_2^2} B(x) \right) - \frac{R_2}{R_1^2 - R_2^2} N(x).$$

The deformation retract of the helix $\alpha_2(t)$ into the retraction $\eta_{11}(\alpha_2(t))$ is defined as

$$\varphi(x, h) = \text{Int}^{(t\rightarrow\infty)} \left( S - \frac{S R^2}{R^2 - R_1^2} \sin\left( S \sqrt{R^2 - R_1^2} \right) + \frac{S R^2}{R^2 - R_1^2} \cos\left( S \sqrt{R^2 - R_1^2} \right) + \frac{R^2}{R_1^2 - R_2^2} T(x) + 2n \cos\left( S \sqrt{R^2 - R_1^2} \right) \right) N(x) +$$

$$+ \left( \frac{S R^2}{R^2 - R_1^2} \sin\left( S \sqrt{R^2 - R_1^2} \right) + \frac{S R^2}{R^2 - R_1^2} \cos\left( S \sqrt{R^2 - R_1^2} \right) + \frac{R^2}{R_1^2 - R_2^2} B(x) \right) - \frac{R_2}{R_1^2 - R_2^2} N(x).$$
Frent frame adapted to is defined the curvature $k_2$ and torsion $k_2$, then provide a complete characterization of the curve. Helix is one of the most fascinating curves in Science and Nature, a helix is a geometry curve with non-vanishing constant curvature $k_2$ and non-vanishing constant torsion $k_2$. A curve of constant slope or general helix in Euclidean 3-space $\mathbb{E}^3$, is defined by the property that the tangent makes a constant angle with a fixed straight line. In the present article, we obtain and study types of retraction of the position vectors of time like helix in Minkowski 3-space. Also, by using the position vectors of the curve and retraction of the position vectors, we deduced types of the deformation retracts of the time like helix in Minkowski 3-space. The relations between the retraction and the deformation retracts of time like helix are obtained. Types of minimal retraction of time like helix in Minkowski 3-space are also presented. New types of homotopy maps are deduced.

References:


