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GAME WITH FUZZY PAYOFFS AND CRISP GAME

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Abstract: In this paper we consider a bi-matrix game with fuzzy payoffs. To compare a fuzzy numbers, some different ordering operators can be used. We define Nash equilibrium in fuzzy game using the ordering operator. The game with crisp payoffs is associated with the original game. Here, crisp payoffs are the operator's value on a fuzzy payoff. We propose the following statement: if the ordering operator is linear, then the game with payoffs has same Nash equilibrium strategy profile as the crisp game. We present an algorithm for constructing a Nash equilibrium in a bi-matrix game with fuzzy payoffs and we are using this fact. We use such ordering operators, and construct the Nash equilibrium in examples of bi-matrix games.

Key Word: Fuzzy set, bi-matrix, membership function, cooperative game

1. Introduction: Game theory takes an important role in decision making and actively is used for modeling the real world. Applying game theory in real situations, it is difficult to have strict value of payoffs, because players are not able to analyze some data of game and as a result, their information isn't complete. This lack of precision and certainty may be modeled by different ways such as fuzzy games. Initially, fuzzy sets were used by [Butnariu, 1978] in non-cooperative game theory. He used fuzzy sets to represent the belief of each player for strategies of other players. Since then, fuzzy set theory has been used in many non-cooperative and cooperative games.

Overview of the results of fuzzy games are in [Larbani, 2009]. In recent past, various attempt have been made in fuzzy bi-matrix game theory namely [Maeda, 2003], [Nayak, 2009], [Dutta, 2014], [*Seikh et al.*, 2015]

In this paper, we present an approach that generalizes some other ideas ([Campos, 1989], [Cunlin, 2011], [Dutta, 2014] *et al.).*

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2. Fuzzy Numbers

Definition 2.1 A fuzzy set is defined as a subset A of universal set $X \subseteq R$ by its membership function $\mu_A(\cdot)$ with assigns to each element $x \in R$, a real number $\mu_A(x)$ in the interval [0, 1].

Definition 2.2 A fuzzy subset a defined on R, is said to be a fuzzy number if its membership function $\mu_A(x)$ satisfies the following conditions:

(1) $\mu_{\tilde{A}}(x): R \to [0,1]$ *is upper semi-continuous;*

(2) $\mu_{\tilde{A}}(x) = 0$ *outside some interval* $[a,d]$;

(3) *There exist real numbers b, c such that* $a \leq b \leq c \leq d$ *and*

(*a*) $\mu_{\tilde{A}}(x)$ *is monotonic increasing on* $[a,b]$;

(*b*) $\mu_{\tilde{A}}(x)$ *is monotonic decreasing on* $[c,d]$;

(*c*) $\mu_{\tilde{A}}(x) = 1, \ \forall \ x \in [b,c]$,

The α-cut of a fuzzy number $\tilde{A}$ plays an important role in parametric ordering of fuzzy numbers. The α-cut or α-level set of a fuzzy number $\tilde{A}$, denoted by $\tilde{A}_\alpha$, is defined as $\tilde{A}_\alpha = \{x \in R : \mu_{\tilde{A}}(x) \geq \alpha\}$ for all $\alpha \in (0,1]$.

The support or 0-cut $\tilde{A}_0$ is defined as the closure of the set

$$\tilde{A}_0 = \{x \in R : \mu_{\tilde{A}}(x) > 0\}$$

Every □-cut is a closed interval $\tilde{A}_\alpha = [g_{\tilde{A}}(\alpha), G_{\tilde{A}}(\alpha)] \subset R$

Where $g_{\tilde{A}}(\alpha) = \inf\{x \in R : \mu_{\tilde{A}}(x) \geq \alpha$ and $G_{\tilde{A}}(\alpha) = \sup\{x \in R : \mu_{\tilde{A}}(x) \geq \alpha\}$

For any $\alpha \in [0,1]$.

We denote the sets of fuzzy number as F. Next, we use two types of fuzzy numbers.

Definition 2.3 Let $\tilde{A}$ be a fuzzy number. If the membership function of $\tilde{A}$ is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x-a+l}{l} & for\ x \in [a-l, a] \\ \frac{a+r-x}{r} & for\ x \in [a, a+r] \\ 0 & otherwise \end{cases}$$

Where, *a*, *l* and *r* are all real (crisp) numbers, and *l*, *r* are non-negative. Then $\tilde{A}$ is called a triangular fuzzy *number, denoted by* $\tilde{A} = (a, l, r)$

We denote the sets of triangular fuzzy number as F_3.

Definition 2.4 Let $\tilde{A}$ be a fuzzy number. If the membership function of $\tilde{A}$ is given by

$$\begin{cases} \frac{x-a+l}{l} & for\ x \in [a-l, a] \\ 1 & for\ x \in [a, b] \\ \frac{b+r-x}{r} & for\ x \in [b, b+r] \\ 0 & otherwise \end{cases}$$

where, *a*, *b*, *l* and *r* are all real (crisp) numbers, and *l*, *r* are non-negative. Then $\tilde{A}$ is called a trapezoidal fuzzy number, denoted by $\tilde{A} = (a,b,l,r)$. $[a,b]$ is the core of $\tilde{A}$.

We denote the sets of trapezoidal fuzzy number as F_3.

Let $\tilde{A} = a_1, l_1, r_1$ and $\tilde{B} = a_2, l_2, r_2$ be two triangular fuzzy numbers. Then arithmetic operations on $\tilde{A}$ and $\tilde{B}$ are defined as follows:

Addition:

$\tilde{A} + \tilde{B} = a_1 + a_2, l_1 + l_2, r_1 + r_2 = \tilde{C} \in F_3$

Scalar multiplication: $\forall k > 0, k \in R$,

$k\tilde{A} = ka_1, kl_1, kr_1 \qquad k\tilde{A} \in F_3$

Let $\tilde{A} = a_1, b_1, l_1, r_1$, and $\tilde{B} = a_2, b_2, l_2, r_2$, be two trapezoidal fuzzy numbers. Then arithmetic operations on $\tilde{A}$ and $\tilde{B}$ are defined as follows:

Addition:

$\tilde{A} + \tilde{B} = a_1 + a_2, b_1 + b_2, l_1 + l_2, r_1 + r_2 = \tilde{C} \in F_4$

Scalar multiplication: $\forall k > 0, k \in R$,

$k\tilde{A} = ka_1, kb_1, kl_1, kr_1 \qquad k\tilde{A} \in F_4$

:

In general, let $\tilde{A}$ and $\tilde{B}$ be two fuzzy numbers. If

$\tilde{A} + \tilde{B} = \tilde{C}\ , \lambda\tilde{A} = \tilde{D}\quad and\ \lambda = const > 0\ then$

$$\tilde{C}_{\alpha} = [g_{\tilde{A}}(\alpha) + g_{\tilde{B}}(\alpha),\ \ G_{\tilde{A}}(\alpha) + G_{\tilde{B}}(\alpha)]$$

And

$$\tilde{D}_{\alpha} = [\lambda g_{\tilde{A}}(\alpha),\qquad \lambda G_{\tilde{A}}(\alpha)]$$

for any $\alpha \in [0; 1]$.

Comparing of fuzzy numbers is a very important question. Various methods for comparing fuzzy numbers have been proposed. For example, fuzzy numbers can be ranked using the defuzzification methods.

A defuzzification is the process of producing a real (crisp) value corresponding to a fuzzy number. In order to rank fuzzy numbers are using the defuzzification approach, the fuzzy numbers are first defuzzified and then, the obtained crisp numbers are ordered using the order relation of real numbers.

Yager in [Yager, 1981] introduced a function for ranking fuzzy subsets in unit interval, which is based on the integral of mean of the α--cuts. Yager index is

$$Y(\tilde{A}) = \frac{1}{2}\int_0^1 [g_{\tilde{A}}(\alpha) + G_{\tilde{A}}(\alpha)]\ d\alpha$$

Jain in [Jain, 1977], Baldwin and Guild in [Baldwin, 1979] were also suggested methods for ordering fuzzy subsets in the unit interval.

Ibanez and Munoz in [Ibanez, 1989] have developed a subjective approach for ranking fuzzy numbers. In [Ibanez, 1989], Ibanez and Munoz defined the following number as the average index for fuzzy number $\tilde{A}$

$$V_P(\tilde{A}) = \int_Y f_{\tilde{A}}(\alpha)\ dP(\alpha)$$

Where Y is a subset of the unit interval and P is a probability distribution on Y. The definition of $f_{\tilde{A}}$ could be subjective for decision maker.

Ukhobotov in [Ukhobotov, 2016] proposed the ordering operator

$$U(\tilde{A}, v) = \frac{1}{2}\int_0^1 [(1-v)g_{\tilde{A}}(\alpha) + v\, G_{\tilde{A}}(\alpha)]\ d\alpha$$

Where crisp parameter $v \in [0; 1]$. Different v correspond to different behavior of the decision maker.

Some other defuzzification operators were given in [Basiura *et al.*, 2015].

Definition 2.5 Let $\tilde{A}$ and $\tilde{B}$ are a fuzzy numbers, $T : F \to R$ is the operator of defuzzification $(T(\cdot) = Y(\cdot), V_P(\cdot),$ $U(\cdot, v)$ etc.).

We say that $\tilde{B}$ is preferable to $\tilde{A}$ by the defuzzification operator T ($\tilde{A} \preccurlyeq_T \tilde{B}$) if and only if

$$T(\tilde{A}) \leq T(\tilde{B})$$

The order relation $\preccurlyeq_T$ depends on the defuzzification operator T.

Example 2.1.

Let

$\tilde{A}, \tilde{B}, \tilde{C} \in F_3\ ,\ \tilde{A} = (52, 12, 22), \tilde{B} = (57, 32, 14), \tilde{C} = (54, 32, 32)$

$T(\cdot) = U(\cdot, v)$

If $\tilde{X} = (a, l, r) \in F_3$ then

$$U(\tilde{X}, v) = a + \frac{v\, r - (1-v)l}{2}$$

Next, if $v = 0$, then

$U(\tilde{A}, 0) = 46,\ U(\tilde{B}, 0) = 41,\ U(\tilde{C}, 0) = 38$.

if $v = \frac{1}{2}$, then

$U\left(\tilde{A}, \frac{1}{2}\right) = 54.5,\ U\left(\tilde{B}, \frac{1}{2}\right) = 52.5,\ U\left(\tilde{C}, \frac{1}{2}\right) = 54$.

if $v = 1$, then

$U(\tilde{A}, 1) = 58,\ U(\tilde{B}, 1) = 64,\ U(\tilde{C}, 1) = 70$.

$\tilde{C} \preccurlyeq_{U(\cdot,0)} \tilde{B} \preccurlyeq_{U(\cdot,0)} \tilde{A}$

$\tilde{B} \preccurlyeq_{U(\cdot,\frac{1}{2})} \tilde{C} \preccurlyeq_{U(\cdot,\frac{1}{2})} \tilde{A}$

$\tilde{A} \preccurlyeq_{U(\cdot,1)} \tilde{B} \preccurlyeq_{U(\cdot,1)} \tilde{C}$

Definition 2.6 *If* $\forall \tilde{A}, \tilde{B} \in F$ $\forall\ \alpha, \beta = const$
$T(\alpha\tilde{A} + \beta\tilde{B}) = \alpha T(\tilde{A}) + \beta T(\tilde{B});$
then the defuzzification operator T(•) is the linear defuzzification operator.
Clear, Yager index $Y(\ \bullet\ )$ and operator $U(\ \bullet\ , v)$ is linear.

1. Crisp Games

3.1 Non-cooperative N-Person Games

Consider a non-cooperative game of *N* players in the class of pure strategies
$\Gamma = \langle \boldsymbol{N}; \{X_i\}_{i\in N}\ \{f_i(x)\}_{i\in N}\rangle$
where $\boldsymbol{N} = 1, \dots\dots, N$ is the set of players' serial numbers; each player *i* chooses and applies his own pure strategy $x_i \in X_i \subseteq R^{n_i}$, forming no coalition with the others, which induces a strategy profile

$$x = (x_1, \dots, x_N) \in X = \prod_{i\in N} X_i \subset R^n$$

for each $i \in \boldsymbol{N}$, a payoff function $f_i(x)$ is defined on the strategy profile set *X*, which gives the payoff of player $i. f_i(x)$ is payoff function of player $i\ (i \in \boldsymbol{N})$.

In addition, denote $(x \parallel z_i) = (x_1, \dots\dots x_{i-1}, z_i, x_{i+1}, \dots\dots, x_N)$,

Definition 3.1. A strategy profile $x^e = (x_1^e, \dots\dots, x_N^e) \in X$ is called a Nash equilibrium in the game (1) if
max

$$\max_{x_i\in X_i} f_i(x^e \parallel x_i) = f_i(x^e) \qquad (i \in \boldsymbol{N})$$

The set of all $\{x^e\}$ in the game (1) will be designated by X^e.

3.2 Bimatrix Games: We consider a bi-matrix game defined by a pair (*A*, *B*) of real $m \times n$ matrices. Matrices *A* and *B* are payoffs to play *I* and *II*, respectively. The set of pure strategies of player *I* (matrix rows) is denoted by *M* and the set of pure strategies of player *II* (columns) is denoted by *N*.
$M = (1, \dots\dots, m), \quad N = (1, \dots\dots, n)$
$(n = \sum_{i\in N} n_i)$

The sets of mixed strategies of the two players are called *X* and *Y* . For mixed strategies *x* and *y*, we want to write expected payoffs as matrix products *xAy* and *xBy*, so that *x* should be a row vector and *y* should be a column vector. That is,

$$X = \{(x_1, \dots, x_m) \mid x_i \geq 0\ (\forall i \in M)\}, \quad \sum_{i\in M} x_i = 1$$

And

$$Y = \{(y_1, \dots, y_n) \mid y_j \geq 0\ (\forall j \in N)\}, \quad \sum_{j\in N} y_j = 1$$

Definition 3.2. *A pair* $(x^e, y^e) \in X \times Y$ *is called a Nash equilibrium for the game* (*A*, *B*) *if* (1)
$x^e A y^e > x A y^e \quad \forall x \in X,$
$x^e B y^e > x^e B y \quad \forall y \in Y.$
From [Nash, 1950] implies that the set of Nash equilibrium for a game (*A;B*) is non-empty.
We recall, a bi-matrix game is a zero-sum bi-matrix game if matrix $B = -A$. A solve of a zero-sum bi-matrix game is saddle-point.

2. Game with Fuzzy Payoffs

Further, we consider a non-cooperative *N*-person game

$$\tilde{\Gamma} = \left\langle \boldsymbol{N}, \{X_i\}_{i\in N}, \{\tilde{f}_i(x)\}_{i\in N} \right\rangle$$

Which differs from (1) only payoffs functions. In (3), a payoff function of player *i* is $\tilde{f}_i(x) : X \to \tilde{F}$. $\tilde{f} = (\tilde{f}_1, \dots, \tilde{f}_N)$.
In addition, X_i contains only a finite number of elements. $\tilde{\Gamma}$ is a finite game with fuzzy payoffs.
If $\boldsymbol{N}' = \{1, 2\}$, then (3) is a bi-matrix game with fuzzy payoffs.$\cdots$ (2)
To determine the concept of optimality, we must compare payoffs. We use some defuzzification operator *T*
$(T(\ \bullet\ ) = Y(\ \bullet\ ), VP(\ \bullet\ ), U(\ \bullet\ ; \_)\ etc.).$
We propose the following definition.

Definition 4.1. A strategy profile $x^e = (x_1^e, \dots\dots, x_N^e) \in X$ is called a T(•)-Nash equilibrium in the game (3) if
$\tilde{f}_i(x^e \parallel x_i) \preccurlyeq_T \tilde{f}_i(x^e) \qquad (i \in \boldsymbol{N}):$

We note that the solutions, which defined in [Maeda, 2003], [Cunlin, 2011] and [Dutta, 2014], are particular cases of Definition 4.1.
Next, we consider the associated crisp game for (3)

$$\tilde{\Gamma} = \left\langle N, \{X_i\}_{i\in N}, \{T(\tilde{f}_i(x))\}_{i\in N} \right\rangle$$

Theorem 4.1. Let xe is a Nash equilibrium in (4) and T(•) is a linear defuzzification operator, then $\mathbf{x^e}$ is T(•)-Nash equilibrium in a game (3).
For example, we consider one bi-matrix game with a triangular fuzzy payoffs.

Example 4.1. Let $\tilde{A}$ and $\tilde{B}$ are the triangular fuzzy payoff matrixes of the fuzzy bi-matrix game $\tilde{\Gamma}$, given as follows:

$$\tilde{A} = \begin{pmatrix} (20,5,10) & (5,10,5) \\ (5,10,5) & (10,5,5) \end{pmatrix} \quad \tilde{B} = \begin{pmatrix} (10,10,5) & (15,5,10) \\ (10,10,20) & (5,10,5) \end{pmatrix}$$

We use the operator $U(\ \bullet, v)$.
If $v = 0$, then the associated crisp game (4) given as follows

$$A = \begin{pmatrix} 17.5 & 0 \\ 0 & 7.5 \end{pmatrix} \quad B = \begin{pmatrix} 5 & 12.5 \\ 5 & 0 \end{pmatrix}$$

The mixed $U(\ \bullet, 0)$-Nash equilibrium is $x^e = (x_1^e, x_2^e)$ where $x_1^e = \left(\frac{2}{5}, \frac{3}{5}\right)$ $x_2^e = \left(\frac{3}{10}, \frac{7}{10}\right)$

If $v = \frac{1}{2}$ then the associated crisp game (4) given as follows

$$A = \begin{pmatrix} 21.25 & 3.75 \\ 3.75 & 10 \end{pmatrix} \quad B = \begin{pmatrix} 8.75 & 16.25 \\ 12.5 & 3.75 \end{pmatrix}$$

The mixed $U(\ \bullet, \frac{1}{2})$-Nash equilibrium is $x^e = (x_1^e, x_2^e)$ where $x_1^e = \left(\frac{7}{13}, \frac{6}{13}\right)$ $x_2^e = \left(\frac{5}{19}, \frac{14}{19}\right)$

If $v = 1$ then the associated crisp game (4) given as follows

$$A = \begin{pmatrix} 25 & 705 \\ 7.5 & 12.5 \end{pmatrix} \quad B = \begin{pmatrix} 12.5 & 20 \\ 20 & 7.5 \end{pmatrix}$$

The mixed $U(\ \bullet, 1)$-Nash equilibrium is $x^e = (x_1^e, x_2^e)$ where $x_1^e = \left(\frac{5}{8}, \frac{3}{8}\right)$ $x_2^e = \left(\frac{2}{9}, \frac{7}{9}\right)$.

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