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Original Research Article

PRE A*-ALGEBRA FUNCTIONS

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Abstract: In this paper, Pre A*-algebra expression is defined and it is extended to normal forms. Pre A*-algebra functions are represented by these normal forms and proved that this representation is unique. Partial list of the collection of the functions is also established.

Keywords: DNF, CNF, Polynomial, functions.

1. Introduction: Pre A*-algebra is introduced by Rao[5] which is analogous to C-algebra and reduct of A*-algebra. It is studied that their subdirect representations, obtained the results that **2**= {0, 1} and **3**= {0, 1, 2} are the subdirectly irreducible Pre-A*-Algebras and every Pre-A*-algebra can be imbedded in $\mathbf{3}^X$ for some set X. .The growth extends to three Boolean algebras $B(A), \Im_{P(A)}$ and $\Im_{M(A)}$ are isomorphic to each other[11], defined dual ideal, dual prime ideals [12], and proved every prime ideal is maximal ideal for the special cases only [13**]**.Now one more feather is function. Generally, algebraic expressions are formed by the combinations of terms by mathematical symbols and functions are represented by these expressions. Here also expressions are formed by the combinations of literals by $\wedge, \vee, (-)\tilde{}$ Disjunctive normal form is constructed by these expressions. These DNF is used to represent the Pre A*-algebra functions. Conjunctive normal form is the dual of DNF.

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2. Preliminaries: In this section we recall the definition of Pre A*-algebra and some results from [5,6] which will be required later.

Definition 2.1: An algebra $(A, \wedge, \vee, (-)\tilde{}\,)$ where A is non-empty set with 1, $\wedge$, $\vee$ are binary operations and $(-)\tilde{}$ is a unary operation satisfying

(a) $x\tilde{}\tilde{} = x, \quad \forall x \in A$

(b) $x \wedge x = x, \quad \forall x \in A$

(c) $x \wedge y = y \wedge x, \quad \forall x, y \in A$

(d) $(x \wedge y)\tilde{} = x\tilde{} \vee y\tilde{}, \quad \forall x, y \in A$

(e) $x \wedge (y \wedge z) = (x \wedge y) \wedge z, \quad \forall x, y, z \in A$

(f) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \quad \forall x, y, z \in A$

(g) $x \wedge y = x \wedge (x\tilde{} \vee y), \quad \forall x, y \in A$.

is called a Pre A*-algebra

Example 2.2:
3 = {0, 1, 2} with operations $\wedge, \vee, (-)^{\sim}$ defined below is a Pre A*-algebra

$\wedge$	0	1	2
0	0	0	2
1	0	1	2
2	2	2	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

x	$x^{\sim}$
0	1
1	0
2	2

Note 2.3: The elements 0, 1, 2 in the above example satisfy the following laws:
(a) $2^{\sim} = 2$ (b) $1 \wedge x = x$ for all $x \in \mathbf{3}$
(c) $0 \vee x = x$, $\forall\ x \in \mathbf{3}$ (d) $2 \wedge x = 2 \vee x = 2$, $\forall\ x \in \mathbf{3}$.
Example 2.4 = {0, 1} with operations $\wedge, \vee, (-)^{\sim}$ defined below is a Pre A*-algebra.

$\wedge$	0	1
0	0	0
1	0	1

$\vee$	0	1
0	0	1
1	1	1

x	$x^{\sim}$
0	1
1	0

Note 2.5 :
(i)($\mathbf{2}, \wedge, \vee, (-)^{\sim}$) is a Boolean algebra. So every Boolean algebra is a Pre A*- algebra
(ii) The identities 1.1(a) and 1.1(d) imply that the varieties of Pre A*-algebras satisfies all the dual statements of 1.1

Lemma2.6 [5,6]: Every Pre A*-algebra satisfies the following laws
(a) $x \vee 1 = x \vee x^{\sim}$ and $x \wedge 0 = x \wedge x^{\sim}$
(b) $x \wedge (x^{\sim} \vee x) = x \vee (x^{\sim} \wedge x) = x$
(c) $(x \vee x^{\sim}) \wedge y = (x \wedge y) \vee (x^{\sim} \wedge y)$
(d) $(x \vee y) \wedge z = (x \wedge z) \vee (x^{\sim} \wedge y \wedge z)$
(e) $x \wedge y = 0$, $x \vee y = 1$, then $y = x^{\sim}$
(f) If $x \vee y = 0$, then $x = y = 0$ and $x \vee y = 1$, then $x \vee x^{\sim} = 1$
(g)If $x \wedge y = 0$, $x \vee y = 1$, then $y = x^{\sim}$

Definition 2.7[5]: Let A be a Pre A*-algebra. An element $x \in A$ is called central element of A if $x \vee x^{\sim} = 1$ and the set $\{x \in A / x \vee x^{\sim} = 1\}$ of all central elements of A is called the centre of A and it is denoted by $B(A)$.

Theorem 2.8[5]: Let A be a Pre A*-algebra with 1, then $B(A)$ is a Boolean algebra with the induced operations $\wedge, \vee, (-)^{\sim}$

Definition2.9 [8]: Let A be a Pre A*-algebra. If $x, p, q \in A$, define $\Gamma_x(p,q) = (x \wedge p) \vee (x^{\sim} \wedge q)$ ($\Gamma_x(p,q)$ should be viewed as a conditional "if x, then p, else q").

Lemma2.10 [8]:
Every Pre A*-algebra with the indicated constants satisfies the following laws
(i) $\Gamma_2(p,q) = 2$ (ii) $\Gamma_x(2,2) = 2$ (iii) $\Gamma_1(p,q) = p$ (iv) $\Gamma_0(p,q) = q$
(v) $\Gamma_x(1,0) = x$

Lemma2.11 [8]:
Every Pre A*-algebra satisfies the laws:
(a) $\Gamma_x(p,q)^{\sim} = \Gamma_x(p^{\sim}, q^{\sim})$
(b)) $\Gamma_x(p,q) \wedge r = \Gamma_x(p \wedge r,\ q \wedge r)$
(c) $\Gamma_x(p,q) \vee \mathrm{r} = \Gamma_x(p \vee \mathrm{r},\ q \vee \mathrm{r})$
(d)
$\Gamma_x(\Gamma_y(p,q), \Gamma_y(r,s)) = \Gamma_y(\Gamma_x(p,r), \Gamma_x(q,s))$

Lemma2.12[8]:
Every Pre A*-algebra with the indicated constants satisfies the following laws.
(a) $\Gamma_{x \vee y}(p,q) = \Gamma_x(p\ , \Gamma_y(p,q))$
(b) $\Gamma_{x \wedge y}(p,q) = \Gamma_x(\ \Gamma_y((p,q),q))$
(c) $\Gamma_p(p,p) = p$

3. Disjunctive Normal Form

Definition 3.1: Let $(A, \wedge, \vee, \sim)$ be a Pre A*-algebra. Expressions involving members of A and the operations $\wedge, \vee, \sim$ are called Pre A*-algebra expressions (or) polynomials.
For example $x \vee y^{\sim}, x, x \wedge 0, etc.,$ are all expressions. Any function specifying these expressions is called functions. i.e., $f(x,y) = x \wedge y$ then f is Pre A*-algebra function and $x \wedge y$ is an expression.

Definition 3.2: A Pre A*-algebra function is said to be in disjunctive normal form in n variables $x_1, x_2, x_3, \ldots\ldots\ldots\ldots x_n$ if it can be written as join of terms of the type $f_1(x_1) \wedge f_2(x_2) \wedge \ldots\ldots\ldots f_n(x_n)$ where $f_i(x_i) = x_i\ or\ x_i^{\sim}\ \ \forall i = 1\ to\ n$ and no two terms are same. $f_1(x_1) \wedge f_2(x_2) \wedge \ldots\ldots\ldots f_n(x_n)$ are called minterms or minimal polynomials.
Thus a minterm in n variables is a product of n literals in which each variable is represented by the variable itself or its complement.

If we have three variables x, y, z then any function in the DNF is

$x \wedge y \wedge z, x^{\sim} \wedge y \wedge z, x \wedge y^{\sim} \wedge z, x \wedge y \wedge z^{\sim},$

$x \wedge y^{\sim} \wedge z^{\sim}, x^{\sim} \wedge y \wedge z^{\sim}, x^{\sim} \wedge y^{\sim} \wedge z, x^{\sim} \wedge y^{\sim} \wedge z^{\sim}.$

Disjunctive Normal Form is in the minimum number of variables.

Theorem 3.3: Every Pre A*-algebra function can be put in DNF.

Proof: It is obvious from Demorgan's law, idempotency, distributive law every Pre A*-algebra function can be put in DNF.

3.1 Conjunctive Normal Form

Definition 3.1.1: A Pre A*-algebra function is said to be in conjunctive normal form in n variables $x_1, x_2, x_3, ...x_n$ if it can be written as meet of terms of the type $f_1(x_1) \vee f_2(x_2) \vee ....f_n(x_n)$ where $f_i(x_i) = x_i \; or \; x_i^{\sim} \;\; \forall i = 1 \; to \; n$ and no two terms are same. $f_1(x_1) \vee f_2(x_2) \vee ....f_n(x_n)$ are called maxterms or maximal polynomials

Theorem 3.1.2: If $f = f_1 \vee f_2........\vee f_n$ be a polynomial in n variables $x_1, x_2, x_3, ...x_n$ in DNF where f_i are minterms then prove that $f_1^{\sim} \wedge f_2^{\sim}.........\wedge f_n^{\sim}$ is the CNF of $f^{\sim}$

Proof: $f = f_1 \vee f_2........\vee f_n$ and $f^{\sim} = f_1^{\sim} \wedge f_2^{\sim}.........\wedge f_n^{\sim}$ each f_i being minterm is of the form $m_1 \wedge m_2..........\wedge m_n$ when each m is x_i or $x_i^{\sim}$ .Thus $f_i^{\sim} = m_1^{\sim} \vee m_2^{\sim}.....\vee m_n^{\sim}$ where each $m_i^{\sim}$ is $x_i \; or \; x_i^{\sim} \;\; \forall i = 1 \; to \; n$ therefore $f_i^{\sim}$ is a maxterm .Hence $f_i^{\sim}$ is a dual of f_i

Remark 3.1.3: CNF is the dual of DNF.

4. Pre A*-algebra Function

In general, a Pre A*-algebra polynomial is any expression which can be built up recursively by repeated application of operations $\wedge, \vee, \sim$ to some set of symbols $x_1, x_2, x_3..............x_n$.

$$\text{Pre A}*\_\text{algebra Polynomials} = < letter > /$$

$$< \text{Pre A}* - \text{algebraPolynomials} \wedge \text{Pre A}* - \text{algebra Polynomials} > /$$

$$< \text{Pre A}* - \text{algebra Polynomials} \vee \text{Pre A}* - \text{algebra Polynomials} > /$$

$$< \text{Pre A}* - \text{algebra Polynomials}^{\sim} >$$

This is the production rule for writing **Pre** A*-algebra Polynomials as algebraic expression. Clearly any polynomial $f(x_1, x_2, x_3..............x_n)$ defines a function $f : A^n \rightarrow A$ on any Pre A*-algebra A. This can be computed by evaluating the polynomial.

Theorem 4.1: Each Pre A*-algebra function can be put into Canonical form in one and only one way.

Proof: Suppose $F = A_1 \vee A_2 \vee A_3...........\vee A_n$, $F = B_1 \vee B_2 \vee B_3...........\vee B_n$ where A_i & B_i are minterms and two representation of a function in canonical form. Now

$$F = A_1 \vee A_2 \vee A_3.............\vee A_n = B_1 \vee B_2 \vee B_3.............\vee B_n$$

$$A_i \leq B_1 \vee B_2 \vee B_3.............\vee B_n \quad \forall i = 1, 2, ...n$$

$$A_i = A_i \wedge (B_1 \vee B_2 \vee B_3.............\vee B_n)$$

$$A_i = (A_i \wedge B_1) \vee (A_i \wedge B_2) \vee (A_i \wedge B_3).............\vee (A_i \wedge B_n)$$

If A_i does not equal any of $B_1, B_2, B_3...........B_n$ then RHS is zero which means A_i =0 but that is not true. Thus A_i equals some B_j .Similarly each B_j equal to some A_i .Hence two representation of functions are same. Therefore there is one and only way to write functions in the canonical form.

Corollary 4.2: Two functions are equal if their respective DNF contain same terms.

Definition 4.3: A Pre A*-algebra function of n variables is a function on A^n into A, where A is the set {0,1,2}, n is a positive integer, and A^n

denotes the n-fold Cartesian product of the set A with itself.

A point $X^* = (x_1, x_2, ...x_n) \in A^n$ is a true point if $f(X^*) = 1$

A point $X^* = (x_1, x_2, ...x_n) \in A^n$ is a false point if $f(X^*) = 0$

A point $X^* = (x_1, x_2, ...x_n) \in A^n$ is a neutral (neither true nor false) point if $f(X^*) = 2$

We denote by 1_n the function which takes constant value 1 on A^n .

We denote by 0_n the function which takes constant value 0 on A^n

We denote by 2_n the function which takes constant value 2 on A^n .

This is about Pre A*-algebra functions

i.e., $f : A^n \to A$

Here f maps each length –n vector, or string, into a single value, or bit.

Hereafter we can choose A^1 as Pre A*-algebra 3^1. *i.e.*, $f : 3^n \to 3$.

Let F_1 consists of all the possible functions $f : \{0,1,2\} \to \{0,1,2\}$. This is the set of all functions $f : 3^1 \to 3$; it is itself the Pre A*-algebra. At each $x \in X = 3^1$ of the domain, $f^{\sim}$ is defined as

$$f^{\sim}(x) = \begin{cases} 1 - f(x) & if \quad x = 0 \ or \ 1 \\ f(x) & if \quad x = 2 \end{cases}$$

where the minus sign has its usual meaning.

The degree of F_1 is 1. The degree of F_2 is 2 and so on.

Let F_2 consists of all the possible functions $f : \{0,1,2\}^2 \to \{0,1,2\}$. This is the set of all functions $f : 3^2 \to 3$.

Table-1-Degree and number of functions

Degree	Number of functions
1	$3^{3^1} = 27$
2	$3^{3^2} = 19,683$
3	$3^{3^3} = 76,25,59,74,84,987$

One example of complement of function and two examples of join and meet of the functions are given in the partial list of elements of F_1. The remaining elements are constant functions 0,1 and 2, and if-then-else of the functions and its complement.

Table-2 Partial list of F_1

	f	$f^{\sim}$	$f \wedge f^{\sim}$	$f \vee f^{\sim}$	$(f \wedge f^{\sim}) \vee (f \vee f^{\sim})$	$(f \wedge f^{\sim}) \wedge (f \vee f^{\sim})$	0	1	2
0	0	1	0	1	1	0	0	1	2
1	1	0	0	1	1	0	0	1	2
2	2	2	2	2	2	2	0	1	2

	$\Gamma_x(f(0), f(0))$	$\Gamma_x(f(0), f(1))$	$\Gamma_x(f(0), f(2))$	$\Gamma_x(f(1), f(0))$	$\Gamma_x(f(1), f(1))$	$\Gamma_x(f(1), f(2))$
$x = 0$	0	1	2	0	1	2
$x = 1$	0	0	2	1	1	2
$x = 2$	2	2	2	2	2	2

	$\Gamma_x(f(2), f(0))$	$\Gamma_x(f(2), f(1))$	$\Gamma_x(f(2), f(2))$	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9
$x = 0$	2	2	2	1	0	2	1	0	2	2	2	2
$x = 1$	2	2	2	1	1	2	0	0	2	2	2	2
$x = 2$	2	2	2	2	2	2	2	2	2	2	2	2

Where α_1 is the complement of $\Gamma_x(f(0), f(0))$, α_2 is the complement of $\Gamma_x(f(0), f(1))$, α_3 is the complement of $\Gamma_x(f(0), f(2))$ and so on.

There are just $3^{3^1} = 27$ non equivalent Pre A*-algebra functions these table are rearranged in lexicographic order of the set of function

.Similarly there are exactly $3^{3^2} = 19,683$ different Pre A*-algebra functions.

Theorem 4.4: If $f : A^n \to A$ is a function in CDNF where A={0,1,2},then one and only minterm will be 1, all others being 0 or 2.Also then CDNF will be identically equal to 2.

Proof: In any value of $f(x_{1,} x_{2,} ....x_{n,})$ as join of minterms, only one term will be $1 \wedge 1 \wedge 1 \wedge .... \wedge 1$ and others will have atleast one zero or one two. Thus minterm will be zero or two.

5. Conclusion:

Each expression can be reduced to one and only one DNF .If any term occur more than once, these can be omitted because of idem potency. CNF and DNF are dual to each other.

DNF and Canonical form represent the same. Functions in the DNF are in the minimum number of variables. The join or meet of a finite number of elements in Pre A*-algebra depends only on the set of elements involved not on the order in which the elements are combined.

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