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Original Research Article

RELAXED MEAN LABELING FOR CORONA OF CYCLE AND STAR GRAPHS

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Abstract: *In this paper, we prove that the graph $C_3\,\hat{o}\,K_{1,m}$ is a relaxed mean graph if and only if $m \le 3$. The graph $C_4\,\hat{o}\,K_{1,m}$ is a relaxed mean graph if and only if $m \le 4$. The graph $C_5\,\hat{o}\,K_{1,m}$ is a relaxed mean graph if and only if $m \le 5$. The graph $C_6\,\hat{o}\,K_{1,m}$ is a relaxed mean graph if and only if $m \le 5$.*

Keywords: Relaxed Mean graph and star.
2010 **Mathematical Subject Classification Number:** $05C78$.

Introduction: In [6,7], the graph $C_n \hat{O} K_{1,m}$ is mean graph if $m = 1,2$. And the graph $C_3 \hat{O} K_{1,m}$ is mean graph if $m = 1,2$. Also, $C_4 \hat{O} K_{1,m}$ is mean graph if $m = 1,2,3$.

Definition 1.1 *The graph $C_n\,\hat{o}\,K_{1,m}$ is the graph obtained by adjoining a vertex of the cycle C_n with the non-pendant vertex of the star $K_{1,m}$.*

Definition 1.2 ***Relaxed Mean Graph*** **[4,5]**
A graph with p vertices and q edges is said to be a relaxed mean graph if there exists a function f from the vertex set of G to $\{0,1,2,\cdots,q-1,q+1\}$ such that the induced map f^* from the edge set of G to $\{1,2,\cdots,q\}$ defined by

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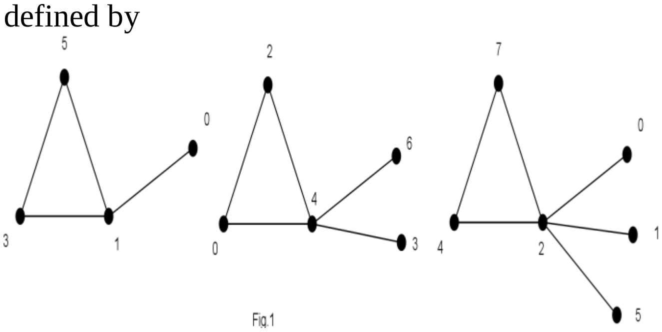

Figure 1: Relaxed Mean Labelled Graph, $C_3 \hat{O} K_{1,m}$, m=1,2,3

$$f^*(e=uv)=\begin{cases}\dfrac{f(u)+f(v)}{2} & \text{if} \quad f(u)+f(v) \text{ is even;}\\[2ex] \dfrac{f(u)+f(v)+1}{2} & \text{if} \quad f(u)+f(v) \text{ is odd}\end{cases}$$

then the resulting edges get distinct labels from the set $\{1,2,\cdots,q\}$.

Theorem 1.3 *The graph $C_3 \hat{o} K_{1,m}$ is a relaxed mean graph if and only if $m \le 3$.*

Proof. Consider the first section of the theorem, the graph $C_3 \hat{o} K_{1,m}$ is a relaxed mean graph if and only if $m \le 3$.

The proof for this section contains two parts, primarily let us prove that the graph $C_3 \hat{o} K_{1,m}$, is a relaxed mean graph when $m = 1,2,3$. The proof to this part is given in the figure 1. Now let us consider the converse part, the contrapositive statement says, $C_3 \hat{o} K_{1,m}$ is not a relaxed mean graph if $m > 3$.

Let $G = C_3 \hat{o} K_{1,m}$, we are supposed to consider the cases when $m > 3$. So considering the primary case, $G = C_3 \hat{o} K_{1,4}$. The vertex and edge set of G is given by,

$$V(G) = \{u,v,w\} \cup \{u_i : 1 \le i \le 4\}$$

$$E(G) = \{uv, vw, wu\} \cup \{uu_i : 1 \le i \le 4\}$$

Then G has 7 vertices and 7 edges. From the definition of relaxed mean labeling the vertex and edge labeling of G is given by

$$f : V(G) \to \{0,1,\cdots,6,8\} \quad \&$$

$$f^* : E(G) \to \{1,2,\cdots 7\}.$$

u is the intersecting or corona vertex of cycle and star graph. The degree of u in G is 6, then the vertex label assigned to u should generate 6 distinct edge labels. So let us search an appropriate vertex label to u from all possible vertex labels.

1. **Claim:** $f(u) \ne 8$.

Suppose if $f(u) = 8$, then if the adjacent vertex takes the minimum possible label, that is 0, the incident edge will get the edge label 4. If the adjacent vertex is assigned the maximum possible label, 6, then that incident edge will get the label 7. Therefore the overall possible edge labels that 8 can generate are $4,5,6,7$. That is $f(u) = 8$ generates only four edge labels which is not sufficient to label 6 edges distinctly. Hence $f(u) \ne 8$.

2. **Claim:** $f(u) \ne 6$.

Suppose if $f(u) = 6$, all possible edge labels 6 is capable of generating are 3,4,5,6,7. That is 6 can induce only 5 distinct edge labels but we require 6 distint edge labels to the 6 edges incident with u. Hence, $f(u) \ne 6$.

3. **Claim:** $f(u) \ne 5$.

Suppose if $f(u) = 5$, all possible edge labels 5 is capable of generating are 3,4,5,6,7. That is 5 can induce only 5 distinct edge labels but we require 6 distint edge labels to the 6 edges incident with u. Hence, $f(u) \ne 5$.

4. **Claim:** $f(u) \ne 4$.

Suppose if $f(u) = 4$, all possible edge labels 4 is capable of generating are 2,3,4,5,6. That is 4 can induce only 5 distinct edge labels but we require 6 distint edge labels to the 6 edges incident with u. Hence, $f(u) \ne 4$.

5. **Claim:** $f(u) \ne 3$.

Suppose if $f(u) = 3$, all possible edge labels 3 is capable of generating are 2,3,4,5,6. That is 3 can induce only 5 distinct edge labels but we require 6 distint edge labels to the 6 edges incident with u. Hence, $f(u) \ne 3$.

6. **Claim:** $f(u) \ne 2$.

Suppose if $f(u) = 2$, all possible edge labels 2 is capable of generating are 1,2,3,4,5. That is 2 can induce only 5 distinct edge labels but we require 6 distint edge labels to the 6 edges incident with u. Hence, $f(u) \ne 2$.

7. **Claim:** $f(u) \ne 1$.

Suppose if $f(u) = 1$, all possible edge labels 1 is capable of generating are 1,2,3,4,5. That is 1 can induce only 5 distinct edge labels but we require 6 distint edge labels to the 6 edges incident with u. Hence, $f(u) \ne 1$.

8. **Claim:** $f(u) \ne 0$.

Suppose if $f(u)=0$, all possible edge labels 0 is capable of generating are 1,2,3,4. That is 0 can induce only 4 distinct edge labels but we require 6 distint edge labels to the 6 edges incident with u. Hence, $f(u)\neq 0$.

Hence $G = C_3\ \hat{o}\ K_{1,4}$ is not a relaxed mean graph.

Also if m takes bigger values then degree of u will also be increasing with respect to m, then the possibilities of edge labels incident with u will be lesser than the nessicity. Thus $G = C_3\ \hat{o}\ K_{1,m}$ is not a relaxed mean graph when $m>3$.

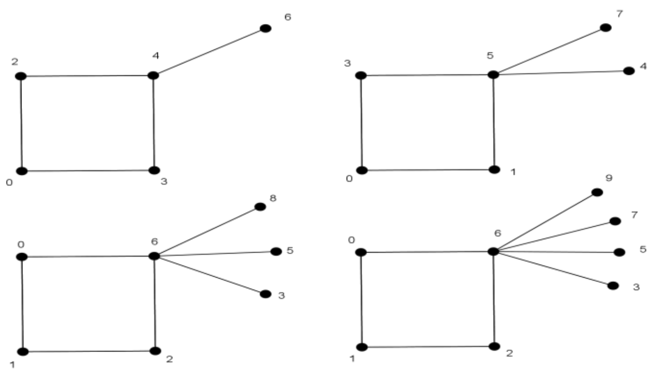

Figure 2: Relaxed Mean Labelled Graph, $C_4\hat{O}K_{1,m}$, m=1,2,3,4

Theorem 1.4 *The graph $C_4\ \hat{o}\ K_{1,m}$ is a relaxed mean graph if and only if $m\leq 4$.*

Proof. The graph $C_4\ \hat{o}\ K_{1,m}$ is a relaxed mean graph if and only if $m\leq 4$. The relaxed mean labeling of the graph $C_4\ \hat{o}\ K_{1,m}$, when $m=1,2,3,4$ is given in the figure 2.

Next it remains to prove that $C_4\ \hat{o}\ K_{1,m}$ is not a relaxed mean graph if $m>4$. Let $G = C_4\ \hat{o}\ K_{1,m}$, we are supposed to consider the cases when $m>4$. So considering the primary case, $G = C_4\ \hat{o}\ K_{1,5}$. The vertex and edge set of G is given by,

$$V(G)=\{u,v,w,x\}\cup\{u_i : 1\leq i\leq 5\}$$
$$E(G)=\{uv,vw,wx,xu\}\cup\{uu_i : 1\leq i\leq 5\}$$

Then G has 9 vertices and 9 edges. From the definition of relaxed mean labeling the vertex and edge labeling of G is given by

$$f : V(G)\rightarrow\{0,1,\cdots,8,10\} \quad \&$$
$$f^* : E(G)\rightarrow\{1,2,\cdots 9\}.$$

u is the intersecting or corona vertex of cycle and star graph. The degree of u in G is 7, then the vertex label assigned to u should generate 7 distinct edge labels. So let us search an appropriate vertex label to u from all possible vertex labels. That is to prove $G = C_4\ \hat{o}\ K_{1,5}$ is not a relaxed mean graph, we have to prove that u cannot be assigned any possible label.

1. **Claim:** $f(u)\neq 10$.

Suppose if $f(u)=10$, then if the adjacent vertex takes the minimum possible label, that is 0, the incident edge will get the edge label 5. If the adjacent vertex is assigned the maximum possible label 8, then that incident edge will get the label 9. Therefore the overall possible edge labels that 10 can generate are $5,6,7,8,9$. That is $f(u)=10$ generates only 5 edge labels which is not sufficient to label 7 edges distinctly. Hence $f(u)\neq 10$.

2. **Claim:** $f(u)\neq 8$.

Suppose if $f(u)=8$, then if the adjacent vertex takes the minimum possible label, that is 0, the incident edge will get the edge label 4. If the adjacent vertex is assigned the maximum possible label, 10, then that incident edge will get the label 9. Therefore the overall possible edge labels that 8 can generate are $4,5,6,7,8,9$. That is $f(u)=8$ generates only 6 edge labels which is not sufficient to label 7 edges distinctly. Hence $f(u)\neq 8$.

3. **Claim:** $f(u)\neq 7$.

Suppose if $f(u)=7$, all possible edge labels 7 is capable of generating are 4,5,6,7,8,9. That is 7 can induce only 6 distinct edge labels but we require 7 distint edge labels to the 7 edges incident with u. Hence, $f(u)\neq 7$.

4. **Claim:** $f(u)\neq 6$.

Suppose if $f(u)=6$, all possible edge labels 6 is capable of generating are 3,4,5,6,7,8. That is 6 can induce only 6 distinct edge labels but we require 7 distint edge labels to the 7 edges incident with u. Hence, $f(u) \neq 6$.

5. **Claim:** $f(u) \neq 5$.

Suppose if $f(u) = 5$, all possible edge labels 5 is capable of generating are 3,4,5,6,7,8. That is 5 can induce only 6 distinct edge labels but we require 7 distint edge labels to the 7 edges incident with u. Hence, $f(u) \neq 5$.

6. **Claim:** $f(u) \neq 4$.

Suppose if $f(u) = 4$, all possible edge labels 4 is capable of generating are 2,3,4,5,6,7. That is 4 can induce only 6 distinct edge labels but we require 7 distint edge labels to the 7 edges incident with u. Hence, $f(u) \neq 4$.

7. **Claim:** $f(u) \neq 3$.

Suppose if $f(u) = 3$, all possible edge labels 3 is capable of generating are 2,3,4,5,6,7. That is 3 can induce only 6 distinct edge labels but we require 7 distint edge labels to the 7 edges incident with u. Hence, $f(u) \neq 3$.

8. **Claim:** $f(u) \neq 2$.

Suppose if $f(u) = 2$, all possible edge labels 2 is capable of generating are 1,2,3,4,5,6. That is 2 can induce only 6 distinct edge labels but we require 7 distint edge labels to the 7 edges incident with u. Hence, $f(u) \neq 2$.

9. **Claim:** $f(u) \neq 1$.

Suppose if $f(u) = 1$, all possible edge labels 1 is capable of generating are 1,2,3,4,5,6. That is 1 can induce only 6 distinct edge labels but we require 7 distint edge labels to the 7 edges incident with u. Hence, $f(u) \neq 1$.

10. **Claim:** $f(u) \neq 0$.

Suppose if $f(u) = 0$, all possible edge labels 0 is capable of generating are 1,2,3,4,5. That is 0 can induce only 5 distinct edge labels but we require 7 distint edge labels to the 7 edges incident with u. Hence, $f(u) \neq 0$.

Suppose if $f(u) = 0$, all possible edge labels 0 is capable of generating are 1,2,3,4,5. That is 0 can induce only 5 distinct edge labels but we require 7 distint edge labels to the 7 edges incident with u. Hence, $f(u) \neq 0$.

Hence $G = C_4 \,\hat{o}\, K_{1,5}$ is not a relaxed mean graph.

Also if m takes bigger values then degree of u will also be increasing with respect to m, then the possibilities of edge labels incident with u will be lesser than the nessicity. Thus $G = C_4 \,\hat{o}\, K_{1,m}$ is not a relaxed mean graph when $m > 4$.

Corollary 1.5 *$G = C_5 \,\hat{o}\, K_{1,m}$ is relaxed mean graph if and only if $m \leq 5$.*

Proof. Similar proof to pevious theorem proves this corollary.

Theorem 1.6 *$C_6 \,\hat{o}\, K_{1,m}$ is relaxed mean graph if and only if $m \leq 5$.*

Proof. First let us prove that $C_6 \,\hat{o}\, K_{1,m}$ is relaxed mean graph if $m \leq 5$.

Let $G = C_6 \,\hat{o}\, K_{1,m}$ the relaxed mean labeling of G when $m = 1, 2, 3, 4, 5$ is given in the figure 3.

Next, it remains to prove that $G = C_6 \,\hat{o}\, K_{1,m}$ is not a relaxed mean graph when $m > 5$.

Primarily consider $G = C_6 \,\hat{o}\, K_{1,6}$, then G has 12 vertices and 12 edges.

The vertex and edge set of G is given by,

$$V(G) = \{u = v_1, v_i : 2 \leq i \leq 6\} \cup \{u_i : 1 \leq i \leq 6\}$$

$$E(G) = \{v_i v_{i+1}\} \cup \{v_1 v_6\} \cup \{u u_i : 1 \leq i \leq 5\}$$

From the definition of relaxed mean labeling the vertex and edge labeling of G is given by

$$f : V(G) \to \{0, 1, \cdots, 11, 13\} \quad \&$$

$$f^* : E(G) \to \{1, 2, \cdots 12\}.$$

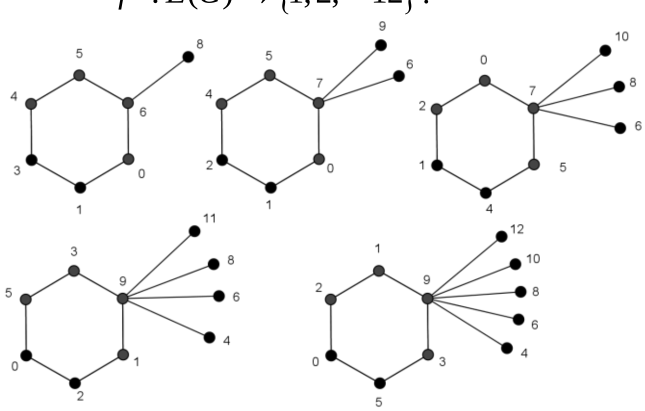

Figure 3: Relaxed Mean Labelled Graph, $C_5 \hat{O} K_{1,m}$, m=1,2,3,4,5

u is the intersecting or corona vertex of cycle and star graph. The degree of u in G is 8, then the vertex label assigned to u should generate 8 distinct edge labels. So let us search an appropriate vertex label to u from all possible vertex labels. That is to prove $G = C_6 \hat{o} K_{1,6}$ is not a relaxed mean graph, we have to prove that u cannot be assigned any possible label.

1. **Claim:** $f(u) \neq 13$.

Suppose if $f(u) = 13$, then if the adjacent vertex takes the minimum possible label, that is 0, the incident edge will get the edge label 7. If the adjacent vertex is assigned the maximum possible label, 11, then that incident edge will get the label 12. Therefore the overall possible edge labels that 13 can generate are $7,8,9,10,11,12$. That is $f(u) = 13$ generates only 6 edge labels which is not sufficient to label 8 edges distinctly. Hence $f(u) \neq 13$.

2. **Claim:** $f(u) \neq 11$.

Suppose if $f(u) = 11$, the possible edge labels that 11 can generate are $6,7,8,9,10,11,12$. That is $f(u) = 11$ generates only 7 edge labels which is not sufficient to label 8 edges distinctly. Hence $f(u) \neq 11$.

3. **Claim:** $f(u) \neq 10$.

Suppose if $f(u) = 10$, possible edge labels that 10 can generate are $5,6,7,8,9,10,11,12$. That is $f(u) = 10$ generates 8 edge labels which is sufficient to label 8 edges distinctly. Also, it is clear that the even labels to $f(u)$ will generate 8 edge labels, before considering these cases we shall see all the cases of odd labels.

4. **Claim:** $f(u) \neq 9$.

Suppose if $f(u) = 9$, the possible edge labels that 9 can generate are $5,6,7,8,9,10,11$. That is $f(u) = 9$ generates only 7 edge labels which is not sufficient to label 8 edges distinctly. Hence $f(u) \neq 9$.

5. **Claim:** $f(u) \neq 7$.

Suppose if $f(u) = 7$, the possible edge labels that 7 can generate are $4,5,6,7,8,9,10$. That is $f(u) = 7$ generates only 7 edge labels which is not sufficient to label 8 edges distinctly. Hence $f(u) \neq 7$.

6. **Claim:** $f(u) \neq 5$.

Suppose if $f(u) = 5$, the possible edge labels that 5 can generate are $3,4,5,6,7,8,9$. That is $f(u) = 5$ generates only 7 edge labels which is not sufficient to label 8 edges distinctly. Hence $f(u) \neq 5$.

7. **Claim:** $f(u) \neq 11$.

Suppose if $f(u) = 3$, the possible edge labels that 3 can generate are $2,3,4,5,6,7,8$. That is $f(u) = 3$ generates only 7 edge labels which is not sufficient to label 8 edges distinctly. Hence $f(u) \neq 3$.

8. **Claim:** $f(u) \neq 11$.

Suppose if $f(u) = 1$, the possible edge labels that 1 can generate are $1,2,3,4,5,6,7$. That is $f(u) = 1$ generates only 7 edge labels which is not sufficient to label 8 edges distinctly. Hence $f(u) \neq 1$.

9. **Claim:** $f(u) \neq 0$.

Suppose if $f(u) = 0$, the possible edge labels that 0 can generate are $1,2,3,4,5,6,7$. That is $f(u) = 0$ generates only 7 edge labels which is not sufficient to label 8 edges distinctly. Hence $f(u) \neq 0$.

Now, let us consider the case when $f(u) = 10$, the vertices adjacent to u are $u_1, u_2, u_3, u_4, u_5, u_6, v_2, v_6$. The edge labels that 10 can generate are $5,6,7,8,9,10,11,12$, if u_i's are given label such that the incident edges receives edge labels, $5,6,7,8,9,10$, then v_2 and v_6 should be allotted labels such that the incident edges receives the edge labels 11 and 12, that is $f(v_2) = 11$ and $f(v_6) = 13$, then the adjacent vertices to v_2 and v_6 should be labelled as such the incident edges generate edge labels less than

5. which is not possible. Therefore the vertices v_2 and v_6 should be allotted smaller labels so that the incident edge will receive smaller edge labels.

We know, $f(u)=10$,

To get the edge label 12, the adjacent vertex should be 13, let $f(u_1)=13$.

To get the edge label 11, the adjacent vertex should be 11, let $f(u_2)=11$.

To get the edge label 10, the adjacent vertex should be 9, let $f(u_3)=9$.

To get the edge label 9, the adjacent vertex shall be 7 or 8, let $f(u_4)=7/8$.

To get the edge label 8, the adjacent vertex shall be 5 or 6, let $f(u_5)=5/6$.

To get the edge label 7, the adjacent vertex shall be 3 or 4, let $f(u_6)=3/4$.

To get the edge label 6, the adjacent vertex shall be 1 or 2, let $f(v_6)=1/2$.

To get the edge label 5, the adjacent vertex should be 0, let $f(v_2)=0$.

To get the edge label 1, the adjacent vertex should be 0 and 1 or 0 and 2. From figure, it is clear that v_2 and v_6 cannot be adjacent. So, the label left over by $f(v_6)$ will be labeled to v_3 to get the edge label 1.

Suppose if $f(v_6)=1$ then $f(v_3)$ will be 2.

To get the edge label 2, $f(v_5)$ should be 3. Note that there is no other choice. Therefore $f(v_5)=3$ implies $f(u_6)=4$.

To get the edge label 4, $f(v_4)$ should be 4 or 5 or 6. But $f(u_6)=4$, so $f(v_4)\neq 4$

If $f(v_4)=5$, then $f^*(v_4v_5)=4$ and $f^*(v_3v_4)=4$, which is a contradiction because all the edges must be mutually exclusive.

If $f(v_4)=6$, then $f^*(v_3v_4)=4$ and $f^*(v_4v_5)=5$, which is a contradiction because we already have $f^*(v_1v_6)=5$.

This contradicts our supposition $f(v_6)=1$. Now let us see what is the case if $f(v_6)=2$.

To get the edge label 2, $f(v_4)$ should be 3. Note that there is no other choice. Therefore $f(v_4)=3$ implies $f(u_6)=4$.

To get the edge label 3, the only significant possibility is the adjacent vertices should be 2 and 3, we already have, $f(v_4)=3$ and $f(v_6)=2$. From figure it is clear that v_4 and v_6 are not adjacent, which contradicts the hypothesis that f^* is a bijection. Hence $G=C_6\,\hat{o}\,K_{1,6}$ is not a relaxed mean graph when $f(u)=10$.

Let us now consider the cases when $f(u)=8,6,4,2$, in general $f(u)=2i$, $i=1,2,3,4$

Observation: Suppose if w and x are adjacent to u and if $f(w)=2n-1$ and $f(x)=2n$, for some n, then

$$f^*(uw)=\frac{2i+2n-1}{2}=i+n=\frac{2i+2n}{2}=f^*(ux)$$

Hence the adjacent vertices of u should be, 0, 1 or 2, 3 or 4, 5 or 6, 7, 9 or 10, 11 and 13. If 0 is labeled to a pendent vertex then there will be no possibility to generate the edge label 1 ($f(u)\neq 2$) and if 13 is labeled to a pendent vertex then there will be no possibility to generate the edge label 12. So the remaining labels 1 or 2, 3 or 4, 5 or 6, 7, 9 or 10, 11 should be placed adjacent to u. To get the edge label 12, the adjacent vertices should be 13 and 10 only (since 11 is placed among the pendent vertices). Note that 13 and 10 should be labeled to the vertices of cycle but any vertex adjacent to 13 will generate the edge label greater than 7 and vertices adjacent to 10 will generate edge label greater than 5. Which will not be exclusive. Hence $f(u)\neq 8,6,4,2$.

Hence $G=C_6\,\hat{o}\,K_{1,6}$ is not a relaxed mean graph for all possible $f(u)$.

Similar argument will prove that $G = C_6 \hat{o} K_{1,m}$ is not a relaxed mean graph for all $m > 5$. Hence $G = C_6 \hat{o} K_{1,m}$ is a relaxed mean graph if and only if $m \leq 5$.

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