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Original Research Article

THE ZEROS OF POLAR DERIVATIVE OF POLYNOMIAL WITH REAL COEFFICIENTS

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Abstract: In this paper we obtain the size of the disc in which the zeros of polar derivatives of polynomial of degree n with real coefficients with respect to a real α lie.

Keywords: Zeros, polar derivatives, polynomials, real α.

Introduction: To estimate the zeros of a polynomial is a long standing problem. It is an interesting area of research for many engineers as well as mathematicians and many results on the topic are available in the literature.

If $P(z)=\sum_{i=0}^{n} a_i z^i$, be a polynomial of degree n then Polar Derivative of the polynomial P(z) with respect to α, where α can be real or complex number, is defined as

$D_\alpha P(z) = n\,P(z) + (\alpha - z)\,P'(z)$.

It is a polynomial of degree up to n-1. The polynomial $D_\alpha P(z)$ generalizes the ordinary derivative, in the sense that $\lim_{\alpha\to\infty} D_\alpha P(z)/\alpha = P'(z)$.

This paper we prove the following results.

Theorem (1): Let $P(z)=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with real coefficients such that for some $\delta \geq 0$

$a_0 + \delta \geq a_1 \geq \dots \geq a_n \geq 0$

and $ia_i \leq (i-1)\, a_{i-1}$ $i= 2, 3, \dots .n$.

Then the zeros of polar derivative of P(z) with respect to a positive α lie in

$|z| \leq (a_{n-1} + \alpha n a_n)^{-1}\{-a_{n-1} - \alpha n a_n + 2na_0 + 2\alpha a_1 + 2n\delta\}$.

Corollary (1): Let $P(z)=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with real coefficients such that for

$a_0 \geq a_1 \geq \dots \geq a_n \geq 0$

and $ia_i \leq (i-1)\, a_{i-1}$ $i= 1, 2, \dots .n$.

Then the polar derivative of P(z) with respect to a positive α has up to (n-1) roots and they lie in

$|z| \leq (a_{n-1} + \alpha n a_n)^{-1}\{-a_{n-1} - \alpha n a_n + 2na_0 + 2\alpha a_1\}$.

Remark(1): By taking δ=0 in Theorem (1) we obtain Corollary(1).

Theorem (2): Let $P(z)=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with real coefficients such that for $\delta \geq 0$

$a_0 + \delta \geq a_1 \geq \dots \geq a_n$

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and $a_i \le (i-1)\, a_{i-1}$ $i = 2, 3, \ldots .n$.
Then the zeros of polar derivative of P(z) with respect to any $\alpha \ne -a_{n-1}/na_n$ lie in
$|z| \le |a_{n-1} + \alpha na_n|^{-1}\{-a_{n-1} - \alpha na_n + 2n\delta + na_0 + \alpha a_1 + | na_0 + \alpha a_1 |\}$.

Corollary(2): Let $P(z)=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with real coefficients such that for
$a_0 \ge a_1 \ge \ldots\ldots \ge a_n$
and $a_i \le (i-1)a_{i-1}$ $i = 1, 2, \ldots .n$.
Then the polar derivative of P(z) with respect to any $\alpha \ne -a_{n-1}/na_n$ has (n-1) roots and they lie in
$|z| \le |a_{n-1} + \alpha na_n|^{-1}\{-a_{n-1} - \alpha na_n + na_0 + \alpha a_1 + | na_0 + \alpha a_1 |\}$.

Remark(2): By taking $\delta=0$ in Theorem (2) we obtain Corollary(2).

Theorem (3): Let $P(z)=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with real coefficients such that for
$a_0 + \delta \ge a_1 \ge \ldots\ldots \ge a_n$
and $a_i \le (i-1)\, a_{i-1}$ $i = 2, 3, \ldots .n$.
Then the polar derivative of P(z) with respect to α such that
$\alpha = -a_{n-1}/ na_n = -2a_{n-2}/(n-1)a_{n-1} = \ldots = -(n-m-1)a_{m+1}/(m+2)a_{m+2} \ne -(n-m)a_m / (m+1)a_{m+1}$
Where m=0,1,..n-1
has exactly m roots and they lie in
$|z| \le |(n-m)a_m + \alpha(m+1)a_{m+1}|^{-1}\{-(n-m)a_m - \alpha(m+1)a_{m+1} + na_0 + \alpha na_1 + 2n\delta + | na_0 + \alpha na_1 |\}$.

Corollay(3): Let $P(z)=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with real coefficients such that for
$a_0 \ge a_1 \ge \ldots\ldots \ge a_n$
and $a_i \le (i-1)\, a_{i-1}$ $i = 1, 2, \ldots .n$.
Then the polar derivative of P(z) with respect to α such that
$\alpha = -a_{n-1}/ na_n = -2a_{n-2}/(n-1)a_{n-1} = \ldots = -(n-m-1)a_{m+1}/(m+2)a_{m+2} \ne -(n-m)a_m / (m+1)a_{m+1}$
where m=0,1,..n-1
has exactly m roots and they lie in
$|z| \le |(n-m)a_m + \alpha(m+1)a_{m+1}|^{-1}\{-(n-m)a_m - \alpha(m+1)a_{m+1} + na_0 + \alpha na_1 + | na_0 + \alpha na_1 |\}$.

Remark(3): By taking $\delta=0$ in Theorem (3) we obtain Corollary(3).

Proof of Theorem 1:
Let $P(z) = a_0 + a_1 z + a_2 z^2 + \ldots\ldots + a_n z^n$ be a polynomial of degree n.
Then the polar derivative of P(z) is given by $D_\alpha P(z) = n\,P(z) + (\alpha - z)\,P'(z)$. Then
$D_\alpha P(z) = [na_0 + \alpha a_1] + [(n-1)a_1 + 2\alpha a_2]\, z + [(n-2)a_2 + 3\alpha a_3]z^2 + \ldots$
$+ [(n-m+1)a_{m-1} + \alpha m a_m]z^{m-1} + [(n-m)a_m + \alpha(m+1)a_{m+1}]z^m + [(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2}]z^{m+1} + \ldots\ldots$
$+ [2a_{n-2} + \alpha(n-1)a_{n-1}]z^{n-2} + [a_{n-1} + \alpha na_n]z^{n-1}$.
Now consider the polynomial $Q(z) = (1-z)\, D_\alpha P(z)$ so that
$Q(z) = -[a_{n-1} + \alpha na_n]z^n + [a_{n-1} + \alpha na_n - 2a_{n-2} - \alpha(n-1)a_{n-1}]z^{n-1} + \ldots$
$+ [(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2} - (n-m)a_m - \alpha(m+1)a_{m+1}]z^{m+1}$
$+ [(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha m a_m]z^m$
$+ [(n-m+1)a_{m-1} + \alpha m a_m - (n-m+2)a_{m-2} - \alpha(m-1)a_{m-1}]z^{m-1} + \ldots$
$+ [(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2]z^2 + [(n-1)a_1 + 2\alpha a_2 - na_0 - \alpha a_1]z$
$+ [na_0 + \alpha a_1]$.
Now if $|z|>1$ then $|z|^{i-n} < 1$ for $i = 1, 2, 3, \ldots..n-1$
Further
$|Q(z)| \ge |a_{n-1} + \alpha na_n||z|^{n-1} - \{|a_{n-1} + \alpha na_n - 2a_{n-2} - \alpha(n-1)a_{n-1}|\, |z|^{n-1} + \ldots.$
$+ |(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2} - (n-m)a_m - \alpha(m+1)a_{m+1}|\, |z|^{m+1}$
$+|(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha m a_m ||z|^m$
$+|(n-m+1)a_{m-1} + \alpha m a_m - (n-m+2)a_{m-2} - \alpha(m-1)a_{m-1}|\, |z|^{m-1}$
$+|(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2|\, |z|^2 +|(n-1)a_1 + 2\alpha a_2 - na_0 - \alpha a_1|\, |z|$
$+|na_0 + \alpha a_1|\}$.
$\ge |a_{n-1} + \alpha na_n|\, |z|^{n-1}[|z| - | a_{n-1} + \alpha na_n|^{-1}\{|a_{n-1} + \alpha na_n - 2a_{n-2} - \alpha(n-1)a_{n-1}|$
$+ |2a_{n-2} + \alpha(n-1)a_{n-1} - 3a_{n-3} - \alpha(n-2)a_{n-2}|\, |z|^{-1} + \ldots$
$+|(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2} - (n-m)a_m - \alpha(m+1)a_{m+1}|\, |z|^{-(n-m-2)}$
$+|(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha m a_m ||z|^{-(n-m-1)}$
$+|(n-m+1)a_{m-1} + \alpha m a_m - (n-m+2)a_{m-2} - \alpha(m-1)a_{m-1}|\, |z|^{-(n-m)} + \ldots$
$+|(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2|\, |z|^{-(n-3)} + |(n-1)a_1 + 2\alpha a_2 - na_0 - \alpha a_1|\, |z|^{-(n-2)} +|na_0 + \alpha a_1|\, |z|^{-(n-1)}$
$\}]$.

$\geq|a_{n-1} + \alpha na_n|\ |z|^{n-1}[|z| - |\ a_{n-1} + \alpha na_n|^{-1}\{|a_{n-1} + \alpha na_n - 2a_{n-2} - \alpha(n-1)a_{n-1}|$
$+ |2a_{n-2} + \alpha(n-1)a_{n-1} - 3a_{n-3} - \alpha(n-2)a_{n-2}| + \ldots$
$+|(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2} - (n-m)a_m - \alpha(m+1)a_{m+1}|$
$+|(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha ma_m|$
$+|(n-m+1)a_{m-1} + \alpha ma_m - (n-m+2)a_{m-2} - \alpha(m-1)a_{m-1}| + \ldots$
$+|(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2| + |(n-1)a_1 + 2\alpha a_2 - na_0 - n\delta - \alpha a_1 + n\delta| + |na_0 + \alpha a_1|\}].$
$\geq|a_{n-1} + \alpha na_n|\ |z|^{n-1}[|z| - |\ a_{n-1} + \alpha na_n|^{-1}\{ 2a_{n-2} + \alpha(n-1)a_{n-1} - a_{n-1} - \alpha na_n$
$+ 3a_{n-3} + \alpha(n-2)a_{n-2} - 2a_{n-2} - \alpha(n-1)a_{n-1} + \ldots$
$+ (n-m)a_m + \alpha(m+1)a_{m+1} - (n-m-1)a_{m+1} - \alpha(m+2)a_{m+2}$
$+ (n-m+1)a_{m-1} + \alpha ma_m - (n-m)a_m - \alpha(m+1)a_{m+1}$
$+ (n-m+2)a_{m-2} + \alpha(m-1)a_{m-1} - (n-m+1)a_{m-1} - \alpha ma_m + \ldots$
$+(n-1)a_1 + 2\alpha a_2 - (n-2)a_2 - 3\alpha a_3 + na_0 + n\delta + \alpha a_1 - (n-1)a_1 - 2\alpha a_2 + n\delta + na_0 + \alpha a_1\}].$
$\geq|a_{n-1} + \alpha na_n|\ |z|^{n-1}[|z| - |\ a_{n-1} + \alpha na_n|^{-1}\{- a_{n-1} - \alpha na_n + 2na_0 + 2n\delta + 2\alpha a_1\}].$
>0 if $|z| > |\ a_{n-1} + \alpha na_n|^{-1}\{- a_{n-1} - \alpha na_n + 2na_0 + 2n\delta + 2\alpha a_1\}$

This shows that if $|z| > 1$ then $Q(z) > 0$ if $|z| > |\ a_{n-1} + \alpha na_n|^{-1}\{- a_{n-1} - \alpha na_n + 2na_0 + 2n\delta + 2\alpha a_1\}$.

Hence all the zeros of Q(z) with $|z|>1$ lie in

$|z| \leq |\ a_{n-1} + \alpha na_n|^{-1}\{- a_{n-1} - \alpha na_n + 2na_0 + 2n\delta + 2\alpha a_1\}$

But those zeros of Q(z) whose modulus is less than or equal to 1, already satisfy the above inequality since all the zeros of $D_\alpha P(z)$ are also the zeros of Q(z) as they lie in the circle defined by the above inequality and this completes the proof.

Proof of Theorem 2:

Let $P(z) = a_0 + a_1z + a_2z^2 + \ldots\ldots + a_nz^n$ be a polynomial of degree n.

Then the polar derivative of P(z) is given by $D_\alpha P(z) = n\,P(z) + (\alpha - z)\,P'(z)$. Then

$D_\alpha P(z) = [na_0 + \alpha a_1] + [(n-1)a_1 + 2\alpha a_2]\,z + [(n-2)a_2 + 3\alpha a_3]z^2 + \ldots$
$+ [(n-m+1)a_{m-1} + \alpha ma_m]z^{m-1} + [(n-m)a_m + \alpha(m+1)a_{m+1}]z^m + [(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2}]z^{m+1} + \ldots\ldots$
$+ [2a_{n-2} + \alpha(n-1)a_{n-1}]z^{n-2} + [a_{n-1} + \alpha na_n]z^{n-1}.$

Now consider the polynomial $Q(z) = (1-z)\,D_\alpha P(z)$ so that

$Q(z) = -[\ a_{n-1} + \alpha na_n]z^n + [a_{n-1} + \alpha na_n - 2a_{n-2} - \alpha(n-1)a_{n-1}]z^{n-1} + \ldots$
$+ [(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2} - (n-m)a_m - \alpha(m+1)a_{m+1}]z^{m+1}$
$+ [(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha ma_m]z^m$
$+ [(n-m+1)a_{m-1} + \alpha ma_m - (n-m+2)a_{m-2} - \alpha(m-1)a_{m-1}]z^{m-1} + \ldots$
$+ [(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2]z^2 + [(n-1)a_1 + 2\alpha a_2 - na_0 - \alpha a_1]z$
$+ [na_0 + \alpha a_1].$

Now if $|z|>1$ then $|z|^{i-n} < 1$ for $i = 1, 2, 3, \ldots\ldots n-1$

Further

$|Q(z)| \geq |a_{n-1} + \alpha na_n||z|^{n-1} - \{|a_{n-1} + \alpha na_n - 2a_{n-2} - \alpha(n-1)a_{n-1}|\ |z|^{n-1} + \ldots.$
$+ |(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2} - (n-m)a_m - \alpha(m+1)a_{m+1}|\ |z|^{m+1}$
$+|(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha ma_m||z|^m$
$+|(n-m+1)a_{m-1} + \alpha ma_m - (n-m+2)a_{m-2} - \alpha(m-1)a_{m-1}|\ |z|^{m-1}$
$+|(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2|\ |z|^2 + |(n-1)a_1 + 2\alpha a_2 - na_0 - \alpha a_1|\ |z|$
$+|na_0 + \alpha a_1|\}.$
$\geq |a_{n-1} + \alpha na_n|\ |z|^{n-1}[|z| - |\ a_{n-1} + \alpha na_n|^{-1}\{|a_{n-1} + \alpha na_n - 2a_{n-2} - \alpha(n-1)a_{n-1}|$
$+ |2a_{n-2} + \alpha(n-1)a_{n-1} - 3a_{n-3} - \alpha(n-2)a_{n-2}|\ |z|^{-1} + \ldots$
$+|(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2} - (n-m)a_m - \alpha(m+1)a_{m+1}|\ |z|^{-(n-m-2)}$
$+|(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha ma_m||z|^{-(n-m-1)}$
$+|(n-m+1)a_{m-1} + \alpha ma_m - (n-m+2)a_{m-2} - \alpha(m-1)a_{m-1}|\ |z|^{-(n-m)} + \ldots$
$+|(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2|\ |z|^{-(n-3)} + |(n-1)a_1 + 2\alpha a_2 - na_0 - \alpha a_1|\ |z|^{-(n-2)} + |na_0 + \alpha a_1|\ |z|^{-(n-1)}\}].$
$\geq|a_{n-1} + \alpha na_n|\ |z|^{n-1}[|z| - |\ a_{n-1} + \alpha na_n|^{-1}\{|a_{n-1} + \alpha na_n - 2a_{n-2} - \alpha(n-1)a_{n-1}|$
$+ |2a_{n-2} + \alpha(n-1)a_{n-1} - 3a_{n-3} - \alpha(n-2)a_{n-2}| + \ldots$
$+|(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2} - (n-m)a_m - \alpha(m+1)a_{m+1}|$
$+|(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha ma_m|$

$+|(n-m+1)a_{m-1} + \alpha m a_m - (n-m+2)a_{m-2} - \alpha(m-1)a_{m-1}| + \ldots$
$+|(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2| + |(n-1)a_1 + 2\alpha a_2 - na_0 - n\delta - \alpha a_1 + n\delta| + |na_0 + \alpha a_1| \}]$.
$\geq |a_{n-1} + \alpha n a_n| |z|^{n-1}[|z| - |a_{n-1} + \alpha n a_n|^{-1}\{ 2a_{n-2} + \alpha(n-1)a_{n-1} - a_{n-1} - \alpha n a_n$
$+ 3a_{n-3} + \alpha(n-2)a_{n-2} - 2a_{n-2} - \alpha(n-1)a_{n-1} + \ldots$
$+ (n-m)a_m + \alpha(m+1)a_{m+1} - (n-m-1)a_{m+1} - \alpha(m+2)a_{m+2}$
$+ (n-m+1)a_{m-1} + \alpha m a_m - (n-m)a_m - \alpha(m+1)a_{m+1}$
$+ (n-m+2)a_{m-2} + \alpha(m-1)a_{m-1} - (n-m+1)a_{m-1} - \alpha m a_m$
$+ \ldots$
$+ (n-1)a_1 + 2\alpha a_2 - (n-2)a_2 - 3\alpha a_3 + na_0 + n\delta + \alpha a_1 - (n-1)a_1 - 2\alpha a_2 + n\delta + |na_0 + \alpha a_1| \}]$.
$\geq |a_{n-1} + \alpha n a_n| |z|^{n-1}[|z| - |a_{n-1} + \alpha n a_n|^{-1}\{- a_{n-1} - \alpha n a_n + na_0 + 2n\delta + \alpha a_1 + |na_0 + \alpha a_1| \}]$.
>0 if $|z| > |a_{n-1} + \alpha n a_n|^{-1}\{- a_{n-1} - \alpha n a_n + na_0 + 2n\delta + \alpha a_1 + |na_0 + \alpha a_1|\}$

This shows that if $|z| > 1$ then $Q(z) > 0$ if $|z| > |a_{n-1} + \alpha n a_n|^{-1} \{- a_{n-1} - \alpha n a_n + na_0 + 2n\delta + \alpha a_1 + |na_0 + \alpha a_1|\}$

Hence all the zeros of $Q(z)$ with $|z|>1$ lie in
$|z| \leq |a_{n-1} + \alpha n a_n|^{-1}\{- a_{n-1} - \alpha n a_n + na_0 + 2n\delta + \alpha a_1 + |na_0 + \alpha a_1| \}$

But those zeros of $Q(z)$ whose modulus is less than or equal to 1, already satisfy the above inequality since all the zeros of $D_\alpha P(z)$ are also the zeros of $Q(z)$ as they lie in the circle defined by the above inequality and this completes the proof.

Proof of Theorem 3:

Let $P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$ be a polynomial of degree n.

Then the polar derivative of $P(z)$ is given by $D_\alpha P(z) = n P(z) + (\alpha - z) P'(z)$. Then

$D_\alpha P(z) = [na_0 + \alpha a_1] + [(n-1)a_1 + 2\alpha a_2] z + [(n-2)a_2 + 3\alpha a_3]z^2 + \ldots$
$+ [(n-m+1)a_{m-1} + \alpha m a_m]z^{m-1} + [(n-m)a_m + \alpha(m+1)a_{m+1}]z^m + [(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2}]z^{m+1} + \ldots + [2a_{n-2} + \alpha(n-1)a_{n-1}]z^{n-2} + [a_{n-1} + \alpha n a_n]z^{n-1}$.

As $\alpha = -a_{n-1}/ na_n = -2a_{n-2}/(n-1)a_{n-1} = \ldots = -(n-m-1)a_{m+1}/(m+2)a_{m+2} \neq -(n-m)a_m / (m+1)a_{m+1}$

$D_\alpha P(z) = [(n-m)a_m + \alpha(m+1)a_{m+1}]z^m + [(n-m+1)a_{m-1} + \alpha m a_m]z^{m-1} + \ldots + [(n-1)a_1 + 2\alpha a_2] z + [na_0 + \alpha a_1]$.

Now consider the polynomial $Q(z) = (1-z) D_\alpha P(z)$ so that

$Q(z) = -[(n-m)a_m + \alpha(m+1)a_{m+1}]z^{m+1} + [(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1}$
$- \alpha m a_m]z^m + [(n-m+1)a_{m-1} + \alpha m a_m - (n-m+2)a_{m-2} - \alpha(m-1)a_{m-1}]z^{m-1} + \ldots$
$+ [(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2]z^2 + [(n-1)a_1 + 2\alpha a_2 - na_0 - \alpha a_1]z$
$+ [na_0 + \alpha a_1]$.

Now if $|z|>1$ then $|z|^{i-m} < 1$ for $i = 1, 2, 3, \ldots m-1$

Further,

$|Q(z)| \geq |(n-m)a_m + \alpha(m+1)a_{m+1}||z|^{m+1} - \{|(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha m a_m| |z|^m$
$+ |(n-m+1)a_{m-1} + \alpha m a_m - (n-m+2)a_{m-2} - \alpha(m-1)a_{m-1}| |z|^{m-1} + \ldots$
$+|(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2| |z|^2 + |(n-1)a_1 + 2\alpha a_2 - na_0 - \alpha a_1| |z|$
$+|na_0 + \alpha a_1|$.
$\geq |(n-m)a_m + \alpha(m+1)a_{m+1}||z|^m [|z| - |(n-m)a_m + \alpha(m+1)a_{m+1}|^{-1}$
$\{|(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha m a_m| + \ldots$
$+|(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2| |z|^{-(m-2)}$
$+|(n-1)a_1 + 2\alpha a_2 - na_0 - \alpha a_1| |z|^{-(m-1)} + |na_0 + \alpha a_1||z|^{-m}\}]$.
$\geq |(n-m)a_m + \alpha(m+1)a_{m+1}||z|^m [|z| - |(n-m)a_m + \alpha(m+1)a_{m+1}|^{-1}$
$\{|(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha m a_m| + \ldots$
$+|(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2|$
$+|(n-1)a_1 + 2\alpha a_2 - na + n\delta - \alpha a_1 - n\delta| + |na_0 + \alpha a_1|\}]$.
$\geq |(n-m)a_m + \alpha(m+1)a_{m+1}||z|^m [|z|$
$- |(n-m)a_m + \alpha(m+1)a_{m+1}|^{-1} \{-(n-m)a_m - \alpha(m+1)a_{m+1}$
$+ (n-m+1)a_{m-1} + \alpha m a_m + \ldots - (n-2)a_2 - 3\alpha a_3 + (n-1)a_1 + 2\alpha a_2$
$-(n-1)a_1 - 2\alpha a_2 + na_0 + \alpha a_1 + 2n\delta + |na_0 + \alpha a_1|\}]$.
$\geq |(n-m)a_m + \alpha(m+1)a_{m+1}||z|^m [|z|$
$- |(n-m)a_m + \alpha(m+1)a_{m+1}|^{-1} \{-(n-m)a_m - \alpha(m+1)a_{m+1} + na_0 + \alpha a_1 + 2n\delta$
$+|na_0 + \alpha a_1|\}]$.
>0 if $|z| > |(n-m)a_m + \alpha(m+1)a_{m+1}|^{-1} \{- (n-m)a_m - \alpha(m+1)a_{m+1} + na_0 + \alpha a_1 + 2n\delta + |na_0 + \alpha a_1|\}$.

This shows that if

$|z| > |(n-m)a_m + \alpha(m+1)a_{m+1}|^{-1} \{-(n-m)a_m - \alpha(m+1)a_{m+1} + na_0 + \alpha a_1 + 2n\delta + |na_0 + \alpha a_1|\}$.

then $Q(z) > 0$.

Hence all the zeros of $Q(z)$ with $|z|>1$ lie in

$|z| > |(n-m)a_m + \alpha(m+1)a_{m+1}|^{-1} \{-(n-m)a_m - \alpha(m+1)a_{m+1} + na_0 + \alpha a_1 + 2n\delta + |na_0 + \alpha a_1|\}$.

But those zeros of $Q(z)$ whose modulus is less than or equal to 1, already satisfy the above inequality since all the zeros of $D_\alpha P(z)$ are also the zeros of $Q(z)$ as they lie in the circle defined by the above inequality and this completes the proof.

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